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On the probabilistic description of a multipartite correlation scenario with arbitrary numbers of settings and outcomes per site

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Abstract

We consistently formalize the probabilistic description of multipartite joint measurements performed on systems of any nature. This allows us (1) to specify in probabilistic terms the difference between nonsignaling, the Einstein–Podolsky–Rosen (EPR) locality and Bell's locality; (2) to introduce the notion of a local hidden variable (LHV) model for an $S_1 \times \cdots \times S_N$-setting N-partite correlation experiment with outcomes of any spectral type, discrete or continuous, and to prove both general and specifically 'quantum' statements on an LHV simulation in an arbitrary multipartite case; (3) to classify LHV models for a multipartite quantum state, in particular, to show that any N-partite quantum state, pure or mixed, admits an arbitrary $S_1 \times 1 \times \cdots \times 1$-setting LHV description; (4) to evaluate a threshold visibility for an arbitrary bipartite noisy quantum state to admit an $S_1 \times S_2$-setting LHV description under any generalized quantum measurements of two parties.

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1. Introduction

The probabilistic description of quantum measurements performed by several parties has been discussed in the literature ever since the seminal publication [1] of Einstein, Podolsky and Rosen (EPR) in 1935. In that paper, the authors argued that *locality*[1] of measurements performed by different parties on perfectly correlated quantum events implies the 'simultaneous reality—and thus definite values'[2] of physical quantities described by noncommuting quantum observables. This EPR argument, contradicting the quantum formalism [2] and referred to as the EPR paradox, seemed to imply a possibility of a *hidden*

[1] In [1], the Einstein–Podolsky–Rosen locality of parties' measurements is otherwise expressed as 'without in any way disturbing' systems observed by other parties.
[2] See [1], page 778.

variable account of quantum measurements. However, the von Neumann 'no-go' theorem [2], published in 1932, was considered wholly to exclude this possibility.

Analysing this problem in 1964–1966, Bell showed [3] that the setting of von Neumann 'no-go' theorem contains the linearity assumption, which is, in general, unjustified, and explicitly constructed [3] the hidden variable (HV) model reproducing the statistical properties of all quantum observables of a qubit system. Considering, however, spin measurements of two parties on a two-qubit quantum system in the singlet state, Bell proved [4] that *any local* hidden variable (LHV) description of these bipartite measurements on perfectly correlated quantum events disagrees with the statistical predictions of quantum theory. Based on his observations in [3, 4], Bell concluded [3] that the EPR paradox should be resolved specifically due to the violation of locality under multipartite quantum measurements and that '. . . non-locality is deeply rooted in quantum mechanics itself and will persist in any completion'[3].

In 1967, Kochen and Specker corrected [6] the setting of von Neumann 'no-go' theorem according to Bell's remark in [3], and proved [6] that, for a quantum system described by a Hilbert space of a dimension $d \geqslant 3$, there does not exist a non-contextual hidden variable (HV) model that reproduces the statistical properties of all quantum observables and conserves the functional subordination between them. Specified for a tensor-product Hilbert space, the Kochen–Specker theorem excludes the existence of the non-contextual HV model for *all* projective measurements on a multipartite quantum state. For *multipartite* projective measurements, this HV model takes the LHV form.

Thus, on one hand, Bell's analysis[4] in [4] does not exclude a possibility for multipartite measurements on an *arbitrary* nonseparable quantum state to admit an LHV model. On the other hand, the Kochen–Specker 'no-go' theorem [6] *does not disprove* the existence for a multipartite quantum state of an LHV model of a *general* type. Therefore, Bell's analysis [4] plus the Kochen–Specker theorem [6] do not disprove that multipartite measurements on an *arbitrary* nonseparable quantum state may admit an LHV model of a general type.

In 1982, Fine [7] formalized the notion of an LHV model for a bipartite correlation experiment (not necessarily quantum), with two settings and two outcomes per site, and proved the main statements on an LHV simulation in this bipartite case.

In 1989, Werner presented [8] the nonseparable bipartite quantum state on $\mathbb{C}^d \otimes \mathbb{C}^d$, $d \geqslant 2$, that admits the LHV model under any bipartite projective measurements performed on this state.

Ever since these seminal publications, the conceptual and mathematical aspects of the LHV description of multipartite quantum measurements have been analysed in plenty of papers, see, for example, [9–15] and references therein. The so-called Bell-type inequalities[5], specifying multipartite measurement situations (correlation experiments) admitting an LHV description, are now widely used in many quantum information tasks.

Nevertheless, as has been recently noted by Gisin [15], in this field, there are still 'many questions, a few answers'.

In our opinion, there is even still a lack in a consistent view on *locality* under multipartite measurements on spatially separated physical systems. For example, Werner and Wolf [11] identify *locality* with *nonsignaling* while Popescu and Rohrlich [10], Barrett, Linden, Massar, Pironio, Popescu and Roberts [13], and Masanes, Acin and Gisin [14] specify quantum multipartite correlations as, in general, *nonlocal* and satisfying '*the no-signaling principle*'.

[3] See [5], page 171.

[4] In the physical literature, Bell's analysis in [4] is referred to as *Bell's theorem*.

[5] A Bell-type inequality represents a linear probabilistic constraint (on either correlation functions or joint probabilities) that holds under any multipartite correlation experiment admitting an LHV description and may be violated otherwise.

In [12], we argue that, in contrast to the opinion of Bell in [3, 5], under a multipartite joint measurement on spacelike separated quantum particles, locality meant by Einstein *et al* in [1], *the EPR locality*, is never violated.

Furthermore, the notion of an LHV model is also understood differently by different authors. For example, for a bipartite quantum state, Werner's notion [8] of an LHV model is not equivalent to that of Fine [7] for bipartite measurements performed on this state.

It should also be stressed that, for an arbitrary multipartite case, there still does not exist either a consistent analysis of a possibility of an LHV simulation or a concise analytical approach to the derivation of extreme Bell-type inequalities for more than two outcomes per site. However, *generalized* bipartite quantum measurements on even two qubits may have infinitely many outcomes.

From the mathematical point of view, the necessity to analyse a possibility of an LHV simulation arises for any multipartite correlation experiment (not necessarily quantum), specified not in terms of a single probability space. The latter is one of the main notions of Kolmogorov's measure–theoretical formulation [16] of probability theory.

The aim of the present paper is to introduce a consistent frame for the probabilistic description of a multipartite correlation experiment on systems of *any* nature and to analyse a possibility of a simulation of such an experiment in LHV terms. The paper is organized as follows.

In sections 2 and 3, we consistently formalize the probabilistic description of multipartite joint measurements with outcomes of any spectral type, discrete or continuous, and specify in probabilistic terms the difference between *nonsignaling* [17], *the EPR locality* [1] and *Bell's locality* [4, 5]. We, in particular, prove (proposition 1) that nonsignaling does not necessarily imply the EPR locality and present the comparative analysis with the specifications of locality and nonsignaling in [10, 11, 13–15]. The details of the probabilistic models for the description of EPR local multipartite joint measurements on physical systems, classical or quantum, are considered in section 3.1.

In section 4, we introduce the notion of an LHV model for an $S_1 \times \cdots \times S_N$-setting N-partite correlation experiment, with outcomes of any spectral type, discrete or continuous, and prove the general statements (theorem 1, proposition 2) on an LHV simulation in an arbitrary multipartite case. An LHV simulation in a general bipartite case and in a dichotomic multipartite case is considered in theorems 2 and 3, respectively.

In section 5, we classify LHV models arising under EPR local multipartite joint measurements on a quantum state. We introduce the notion of an $S_1 \times \cdots \times S_N$-setting LHV description of an N-partite quantum state, prove the main general statements (propositions 3–6) on this notion, and establish its relation to Werner's notion [8] of an LHV model for a multipartite quantum state.

The main results of the present paper are summarized in section 6.

2. Multipartite joint measurements

Consider a measurement situation where each nth of N parties (players) performs a measurement, specified by a setting s_n, and Λ_n is a set of outcomes λ_n, not necessarily real numbers, observed by the nth party (equivalently, at the nth site).

This measurement situation defines the joint[6] measurement with outcomes in $\Lambda_1 \times \cdots \times \Lambda_N$. We call this joint measurement N-partite, and specify it by an N-tuple $(s_1, \ldots, s_N)$ of measurement settings where nth argument refers to a setting at the nth site.

[6] Any measurement with outcomes in a direct product set is called *joint*.

For an N-partite joint measurement $(s_1, \ldots, s_N)$, denote by

$$P_{(s_1,\ldots,s_N)}(D_1 \times \cdots \times D_N) := \operatorname{Prob}\{\lambda_1 \in D_1, \ldots, \lambda_N \in D_N\} \tag{1}$$

the joint probability of events $D_1 \subseteq \Lambda_1, \ldots, D_N \subseteq \Lambda_N$, observed by the corresponding parties and by[7]

$$\langle \Psi(\lambda_1, \ldots, \lambda_N)\rangle := \int \Psi(\lambda_1, \ldots, \lambda_N) P_{(s_1,\ldots,s_N)}(\mathrm{d}\lambda_1 \times \cdots \times \mathrm{d}\lambda_N) \tag{2}$$

the expected value of a bounded measurable real-valued function $\Psi(\lambda_1, \ldots, \lambda_N)$. Specified for a function Ψ of the product form, notation (2) takes the form

$$\langle \varphi_1(\lambda_1) \cdot \ldots \cdot \varphi_N(\lambda_N)\rangle = \int \varphi_1(\lambda_1) \cdot \ldots \cdot \varphi_N(\lambda_N) P_{(s_1,\ldots,s_N)}(\mathrm{d}\lambda_1 \times \cdots \times \mathrm{d}\lambda_N) \tag{3}$$

and may refer either to the joint probability[8]:

$$\begin{aligned} \left\langle \chi_{D_1}(\lambda_1) \cdot \ldots \cdot \chi_{D_N}(\lambda_N)\right\rangle &= \int \chi_{D_1}(\lambda_1) \cdot \ldots \cdot \chi_{D_N}(\lambda_N) P_{(s_1,\ldots,s_N)}(\mathrm{d}\lambda_1 \times \cdots \times \mathrm{d}\lambda_N) \\ &= P_{(s_1,\ldots,s_N)}(D_1 \times \cdots \times D_N), \end{aligned} \tag{4}$$

or, if outcomes are real valued and bounded, to the mean value

$$\left\langle \lambda_{n_1} \cdot \ldots \cdot \lambda_{n_M}\right\rangle = \int \lambda_{n_1} \cdot \ldots \cdot \lambda_{n_M} P_{(s_1,\ldots,s_N)}(\mathrm{d}\lambda_1 \times \cdots \times \mathrm{d}\lambda_N) \tag{5}$$

of the product of outcomes observed at $M \leqslant N$ sites: $1 \leqslant n_1 < \cdots < n_M \leqslant N$. For $M \geqslant 2$, the mean value (5) is referred to as *the correlation function*. A correlation function for an N-partite joint measurement is called *full* whenever $M = N$.

If only outcomes of $M < N$ parties $1 \leqslant n_1 < \cdots < n_M \leqslant N$ are taken into account while outcomes of all other parties are ignored, then the joint probability distribution of outcomes observed at these M sites is given by the following marginal:

$$P_{(s_1,\ldots,s_N)}\big(\Lambda_1 \times \cdots \times \Lambda_{n_1-1} \times \mathrm{d}\lambda_{n_1} \times \Lambda_{n_1+1} \times \cdots \times \Lambda_{n_M-1} \times \mathrm{d}\lambda_{n_M} \times \Lambda_{n_M+1} \times \cdots \times \Lambda_N\big) \tag{6}$$

of distribution $P_{(s_1,\ldots,s_N)}$. In particular, the marginal

$$P_{(s_1,\ldots,s_N)}(\Lambda_1 \times \cdots \times \Lambda_{n-1} \times \mathrm{d}\lambda_n \times \Lambda_{n+1} \times \cdots \times \Lambda_N) \tag{7}$$

represents the probability distribution of outcomes observed at the nth site.

Recall that events $D_1, \ldots, D_N$ observed by N parties are *probabilistically independent* [18] if

$$P_{(s_1,\ldots,s_N)}(D_1 \times \cdots \times D_N) = \prod_n P_{(s_1,\ldots,s_N)}(\Lambda_1 \times \cdots \times \Lambda_{n-1} \times D_n \times \Lambda_{n+1} \times \cdots \times \Lambda_N). \tag{8}$$

3. Nonsignaling, the EPR locality and Bell's locality

Consider now an N-partite measurement situation where any nth party performs $S_n \geqslant 1$ measurements, each specified by a positive integer $s_n \in \{1, \ldots, S_n\}$. Let $\Lambda_n^{(s_n)}$ be a set of outcomes $\lambda_n^{(s_n)}$, observed under s_nth measurement at the nth site.

This measurement situation (N-partite correlation experiment) is described by the whole family

$$\mathcal{E} = \{(s_1, \ldots, s_N) \mid s_1 = 1, \ldots, S_1, \ldots, s_N = 1, \ldots, S_N\}, \tag{9}$$

[7] For an integral over all values of variables, the domain of integration is not usually specified.

[8] Here, $\chi_D(\lambda)$, $\lambda \in \Lambda$ is an indicator function of a subset $D \subseteq \Lambda$. That is, $\chi_D(\lambda) = 1$ if $\lambda \in D$ and $\chi_D(\lambda) = 0$ if $\lambda \notin D$.

consisting of $S_1 \times \cdots \times S_N$ joint measurements $(s_1, \ldots, s_N)$ with joint probability distributions $P^{(\mathcal{E})}_{(s_1,\ldots,s_N)}$ that may, in general, depend not only on settings of the corresponding joint measurement $(s_1, \ldots, s_N)$, but also on a structure of the whole experiment $\mathcal{E}$, in particular, on settings of other parties' measurements.

Let, for any joint measurements $(s_1, \ldots, s_N), (s'_1, \ldots, s'_N) \in \mathcal{E}$, with $M < N$ common settings $s_{n_1}, \ldots, s_{n_M}$ at arbitrary sites $1 \leqslant n_1 < \cdots < n_M \leqslant N$, the marginal probability distributions (6) of outcomes observed at these sites coincide, that is:

$$P^{(\mathcal{E})}_{(s_1,\ldots,s_N)}\big(\Lambda_1^{(s_1)} \times \cdots \times \Lambda_{n_1-1}^{(s_{n_1-1})} \times \mathrm{d}\lambda_{n_1}^{(s_{n_1})} \times \cdots \times \mathrm{d}\lambda_{n_M}^{(s_{n_M})} \times \Lambda_{n_M+1}^{(s_{n_M+1})} \times \cdots \times \Lambda_N^{(s_N)}\big)$$
$$= P^{(\mathcal{E})}_{(s'_1,\ldots,s'_N)}\big(\Lambda_1^{(s'_1)} \times \cdots \times \Lambda_{n_1-1}^{(s'_{n_1-1})} \times \mathrm{d}\lambda_{n_1}^{(s_{n_1})} \times \cdots \times \mathrm{d}\lambda_{n_M}^{(s_{n_M})} \times \Lambda_{n_M+1}^{(s'_{n_M+1})} \times \cdots \times \Lambda_N^{(s'_N)}\big). \tag{10}$$

If parties' measurements are performed on *spatially separated physical systems*, then (10) constitutes a necessary condition for *nonsignaling* in the sense that (i) a measurement device of each party does not directly affect physical systems and measurement devices at other sites; (ii) spatially separated physical systems either do not interact with each other or interact *locally*[9] with interaction signals[10] coming from one system to another already after measurements upon them. If observed physical systems interact during measurements nonlocally, then the nonsignaling condition (10) is, in general, violated.

For a general multipartite correlation experiment, we use a similar terminology.

Definition 1. *For a family (9) of N-partite joint measurements, we refer to (10) as the nonsignaling condition.*

Let further a measurement of each party be *local* in the EPR sense [1]. As specified in footnote 1, the latter means that results of this measurement are not 'in any way disturbed' [1] by measurements performed by other parties.

In probabilistic terms, *the EPR locality* of all parties' measurements under a joint measurement $(s_1, \ldots, s_N) \in \mathcal{E}$ is expressed[11] by the dependence of distribution $P^{(\mathcal{E})}_{(s_1,\ldots,s_N)}$ and all its marginals (6) only on settings of the corresponding measurements at the corresponding sites, that is, by the relation

$$P^{(\mathcal{E})}_{(s_1,\ldots,s_N)}\big(\Lambda_1^{(s_1)} \times \cdots \times \Lambda_{n_1-1}^{(s_{n_1-1})} \times \mathrm{d}\lambda_{n_1}^{(s_{n_1})} \times \cdots \times \mathrm{d}\lambda_{n_M}^{(s_{n_M})} \times \Lambda_{n_M+1}^{(s_{n_M+1})} \times \cdots \times \Lambda_N^{(s_N)}\big)$$
$$\equiv P_{(s_{n_1},\ldots,s_{n_M})}\big(\mathrm{d}\lambda_{n_1}^{(s_{n_1})} \times \cdots \times \mathrm{d}\lambda_{n_M}^{(s_{n_M})}\big), \tag{11}$$

holding for any $1 \leqslant n_1 < \cdots < n_M \leqslant N$ and any $1 \leqslant M \leqslant N$.

With respect to an N-partite joint measurement, relation (11) induces the following general notion.

Definition 2. *An N-partite joint measurement $(s_1, \ldots, s_N) \in \mathcal{E}$ is EPR local if its joint probability distribution has the form $P^{(\mathcal{E})}_{(s_1,\ldots,s_N)} \equiv P_{(s_1,\ldots,s_N)}$ and all marginals of $P_{(s_1,\ldots,s_N)}$ satisfy condition (11).*

Note that condition (11) does not imply the product form of distribution $P_{(s_1,\ldots,s_N)}$. Therefore, under an EPR local multipartite joint measurement, events observed at different sites *do not need* to be probabilistically independent.

[9] In the sense that the physical principle of local action [17] is not violated.
[10] Interaction signals between physical systems cannot propagate faster than light.
[11] For a bipartite case, this definition was introduced in [12].

For an EPR local N-partite joint measurement $(s_1, \ldots, s_N)$, the marginal probability distribution (7) of outcomes observed at the nth site is determined only by a measurement s_n at this site, and we further denote it by

$$P_n^{(s_n)}\big(\mathrm{d}\lambda_n^{(s_n)}\big) := P_{(s_1,\ldots,s_N)}\big(\Lambda_1^{(s_1)} \times \cdots \times \Lambda_{n-1}^{(s_{n-1})} \times \mathrm{d}\lambda_n^{(s_n)} \times \Lambda_{n+1}^{(s_{n+1})} \times \cdots \times \Lambda_N^{(s_N)}\big). \tag{12}$$

From (11) it follows that any family of EPR local N-partite joint measurements satisfies *the nonsignaling condition* (10). However, the converse of this statement is not, in general, true.

Proposition 1. *For a family (9) of N-partite joint measurements satisfying the nonsignaling condition (10), each of joint measurements does not need to be EPR local.*

Proof. Consider, for example, the family $\mathcal{E}' = \{(a_i, b_k) \mid i, k = 1, 2\}$ of bipartite[12] joint measurements, with two settings at each site and the joint probability distributions[13]

$$P_{(a_i,b_k)}^{(\mathcal{E}')}\big(\mathrm{d}\lambda_1^{(a_i)} \times \mathrm{d}\lambda_2^{(b_k)}\big) = \int_\Omega P_1^{(a_i)}\big(\mathrm{d}\lambda_1^{(a_i)}\big|\omega\big) P_2^{(b_k)}\big(\mathrm{d}\lambda_2^{(b_k)}\big|\omega\big)\tau_{a_1,a_2}^{(b_1,b_2)}(\mathrm{d}\omega), \qquad i, k = 1, 2, \tag{13}$$

where measure $\tau_{a_1,a_2}^{(b_1,b_2)}$ depends on all measurements at both parties. From relations

$$\begin{aligned} P_{(a_i,b_1)}^{(\mathcal{E}')}\big(\mathrm{d}\lambda_1^{(a_i)} \times \Lambda_2^{(b_1)}\big) &= P_{(a_i,b_2)}^{(\mathcal{E}')}\big(\mathrm{d}\lambda_1^{(a_i)} \times \Lambda_2^{(b_2)}\big) \\ &= \int_\Omega P_1^{(a_i)}\big(\mathrm{d}\lambda_1^{(a_i)}\big|\omega\big)\tau_{a_1,a_2}^{(b_1,b_2)}(\mathrm{d}\omega), \qquad \forall\, i = 1, 2, \end{aligned} \tag{14}$$

and

$$\begin{aligned} P_{(a_1,b_k)}^{(\mathcal{E}')}\big(\Lambda_1^{(a_1)} \times \mathrm{d}\lambda_2^{(b_k)}\big) &= P_{(a_2,b_k)}^{(\mathcal{E}')}\big(\Lambda_1^{(a_2)} \times \mathrm{d}\lambda_2^{(b_k)}\big) \\ &= \int_\Omega P_2^{(b_k)}\big(\mathrm{d}\lambda_2^{(b_k)}\big|\omega\big)\tau_{a_1,a_2}^{(b_1,b_2)}(\mathrm{d}\omega), \qquad \forall\, k = 1, 2, \end{aligned} \tag{15}$$

it follows that marginals of $P_{(a_i,b_k)}^{(\mathcal{E}')}$, $i, k = 1, 2$, satisfy the nonsignaling condition (10), though do not, in general, need to satisfy the EPR locality condition (11). □

For an N-partite joint measurement $(s_1, \ldots, s_N)$ performed on spatially separated physical systems, the EPR locality corresponds to *nonsignaling plus no feedback* of performed measurements on a state of a composite physical system before all of parties' measurements.

Along with the nonsignaling condition (10) and the EPR locality (11), let us also specify in probabilistic terms the concept of Bell's locality, introduced in [4, 5] for a family of multipartite joint measurements performed on an identically prepared composite physical system consisting of spacelike separated particles. This type of locality corresponds to *nonsignaling plus no feedback plus the existence of variables* $\omega \in \Omega$ of a composite system such that whenever this system is initially characterized by a variable $\omega \in \Omega$ with certainty, then, under each joint measurement $(s_1, \ldots, s_N) \in \mathcal{E}$, any events observed at different sites are *probabilistically independent:*

$$P_{(s_1,\ldots,s_N)}\big(\mathrm{d}\lambda_1^{(s_1)} \times \cdots \times \mathrm{d}\lambda_N^{(s_N)}\big|\omega\big) = P_1^{(s_1)}\big(\mathrm{d}\lambda_1^{(s_1)}\big|\omega\big) \cdot \ldots \cdot P_N^{(s_N)}\big(\mathrm{d}\lambda_N^{(s_N)}\big|\omega\big), \qquad \forall\, \omega \in \Omega. \tag{16}$$

[12] In the quantum information literature, two parties are traditionally named as Alice and Bob and their measurements are usually labeled by a_i and b_k.

[13] This family of bipartite joint measurements was introduced in [12].

If a composite system is initially specified by a probability distribution ν of variables $\omega \in \Omega$ then (16) and the law of total probability[14] imply

$$P_{(s_1,\ldots,s_N)}\left(\mathrm{d}\lambda_1^{(s_1)} \times \cdots \times \mathrm{d}\lambda_N^{(s_N)}\right) = \int_\Omega P_1^{(s_1)}\left(\mathrm{d}\lambda_1^{(s_1)}\middle|\omega\right) \cdot \ldots \cdot P_N^{(s_N)}\left(\mathrm{d}\lambda_N^{(s_N)}\middle|\omega\right)\nu(\mathrm{d}\omega). \tag{17}$$

For a general family of N-partite joint measurements, this concept induces the following notion.

Definition 3. *A family (9) of N-partite joint measurements is Bell local if any of its joint probability distributions admits representation (17) where a probability distribution ν does not depend on performed measurements.*

From (10), (11), (17) and proposition 1 it follows that, for an N-partite correlation experiment,

$$\textit{Bell's locality} \Rightarrow \textit{EPR locality} \Rightarrow \textit{Nonsignaling}. \tag{18}$$

The converse implications are not, in general, true.

Relation (18) between the type of locality meant by EPR in [1] and the type of locality argued by Bell [4, 5] indicates that, in contrast to the opinion of Bell [3, 5], the EPR paradox [1] cannot be, *in principle*, resolved via the violation of Bell's locality. Moreover, as is shown in section 3.1, under a multipartite joint measurement on spacelike separated quantum particles, the EPR locality is not violated.

Let us now analyse the specification of locality and nonsignaling by other authors.

Werner and Wolf [11] identify 'locality' with 'nonsignaling' and define it by the combination of the nonsignaling condition (10) with the EPR locality condition (11), specified for a bipartite case. Thus, Werner–Wolf's locality [11] constitutes the EPR locality.

Popescu–Rohrlich's [10] 'relativistic causality' (nonsignaling) constitutes the EPR locality (11). Barrett–Linden–Massar–Pironio–Popescu–Roberts's [13] 'nonsignaling boxes' correspond to EPR local multipartite correlation experiments. In both papers [10, 13], 'nonlocality' is defined via the violation of a Bell-type inequality (see footnote 5 and section 4). Masanes *et al* [14] and Gisin [15] define 'nonsignaling' and 'nonlocality' similarly to [13].

To our knowledge, the difference (18) between nonsignaling [17], the EPR locality [1] and Bell's locality [4, 5] has not been earlier specified in the literature.

We stress that the so-called 'quantum nonlocality', discussed in the physical literature ever since the seminal publications [3–5] of Bell, does not constitute the violation of locality of quantum interactions—under a multipartite joint measurement on spacelike separated quantum particles, *locality of quantum interactions is not violated* (see in section 3.1).

3.1. EPR local physical models

Consider now the details of the probabilistic models describing *EPR local N*-partite joint measurements, performed on a composite *physical* system, classical or quantum.

3.1.1. EPR local classical model. Let, under an EPR local N-partite joint measurement, each party perform a measurement on a *classical* subsystem. In this case, there always exist variables $\theta \in \Theta$ and a probability distribution π (a classical state) of these variables, characterizing a composite classical system before measurements and such that, for *any EPR*

[14] See, for example, in [18].

local N-partite joint measurement $(s_1, \ldots, s_N)$ on this classical system in a state π, the joint probability distribution $P_{(s_1,\ldots,s_N)}(\cdot|\pi)$ has the form

$$P_{(s_1,\ldots,s_N)}\big(\mathrm{d}\lambda_1^{(s_1)} \times \cdots \times \mathrm{d}\lambda_N^{(s_N)}\big|\pi\big) = \int_\Theta P_1^{(s_1)}\big(\mathrm{d}\lambda_1^{(s_1)}\big|\theta\big) \cdot \ldots \cdot P_N^{(s_N)}\big(\mathrm{d}\lambda_N^{(s_N)}\big|\theta\big)\pi(\mathrm{d}\theta), \tag{19}$$

where, for a variable $\theta \in \Theta$ defined initially with certainty, $P_n^{(s_n)}(\cdot|\theta)$ represents the probability distribution of outcomes observed under s_nth classical measurement at the nth site. In (19), the EPR locality follows from the independence *(no feedback)* of variables θ and a state π on performed measurements *plus* the independence (nonsignaling) of each conditional distribution $P_n^{(s_n)}(\cdot|\theta)$ on measurements of other parties.

Let a classical measurement s_n at the nth site be *ideal*, that is, describe without an error a property of a composite classical system existed before this measurement. On a measurable space[15] $(\Theta, \mathcal{F}_\Theta)$, representing a classical composite system before measurements, any of its observed properties is described by a measurable function $f_{n,s_n} : \Theta \to \Lambda_n^{(s_n)}$. In the ideal case, distribution $P_n^{(s_n,\mathrm{ideal})}(\cdot|\theta)$, standing (19), takes the form

$$P_n^{(s_n,\mathrm{ideal})}\big(D_n^{(s_n)}\big|\theta\big) = \chi_{f_{n,s_n}^{-1}(D_n^{(s_n)})}(\theta), \tag{20}$$

where

$$f_{n,s_n}^{-1}\big(D_n^{(s_n)}\big) = \big\{\theta \in \Theta \big| f_{n,s_n}(\theta) \in D_n^{(s_n)}\big\} \in \mathcal{F}_\Theta \tag{21}$$

is the preimage of a subset $D_n^{(s_n)} \subseteq \Lambda_n^{(s_n)}$ in $\mathcal{F}_\Theta$ under mapping f_{n,s_n}. If classical measurements of all parties are ideal, then substituting (20) into (19), we derive that, under an *ideal classical EPR local N*-partite joint measurement $(s_1, \ldots, s_N)$, the joint probability distribution $P_{(s_1,\ldots,s_N)}^{(\mathrm{ideal})}$ has the image form

$$P_{(s_1,\ldots,s_N)}^{(\mathrm{ideal})}\big(D_1^{(s_1)} \times \cdots \times D_N^{(s_N)}\big|\pi\big) = \pi\big(f_{1,s_1}^{-1}\big(D_1^{(s_1)}\big) \cap \cdots \cap f_{N,s_N}^{-1}\big(D_N^{(s_N)}\big)\big). \tag{22}$$

3.1.2. EPR local quantum model. If an EPR local N-partite joint measurement is performed on a *quantum* N-partite system, then this system is initially specified by a density operator ρ (a quantum state) on a complex separable Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ and, for any *EPR local* N-partite joint measurement performed on this system in a state ρ, the joint probability distribution $P_{(s_1,\ldots,s_N)}(\cdot|\rho)$ is given by

$$P_{(s_1,\ldots,s_N)}\big(\mathrm{d}\lambda_1^{(s_1)} \times \cdots \times \mathrm{d}\lambda_N^{(s_N)}\big|\rho\big) = \mathrm{tr}\big[\rho\big\{M_1^{(s_1)}\big(\mathrm{d}\lambda_1^{(s_1)}\big) \otimes \cdots \otimes M_N^{(s_N)}\big(\mathrm{d}\lambda_N^{(s_N)}\big)\big\}\big], \tag{23}$$

where $M_n^{(s_n)}\big(\mathrm{d}\lambda_n^{(s_n)}\big)$ is a positive operator-valued (POV) measure[16], describing s_nth quantum measurement at the nth site. In (23), the EPR locality is expressed by the independence (*no feedback*) of state ρ on performed measurements *plus* the independence (nonsignaling) of each $M_n^{(s_n)}$ on measurements at other sites.

If the s_nth measurement of the nth party is *ideal,* that is, reproduces without an error a real-valued quantum property described on $\mathcal{H}_n$ by a quantum observable W_{s_n}, then the corresponding POV measure $M_n^{(s_n)}$ is projection valued and is given by the spectral measure $E_{W_{s_n}}$ of observable W_{s_n}.

Let, for example, an N-partite joint measurement be performed on spacelike separated quantum particles in a state ρ on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$. Then its joint probability distribution has the form (23), satisfying the EPR locality condition (11).

Thus, *under any multipartite joint measurement on spacelike separated quantum particles, the EPR locality (hence, nonsignaling) is not violated.*

[15] In this pair, $\mathcal{F}_\Theta$ is a sigma algebra of subsets of a set Θ. For details, see [18, 19].

[16] $M_n^{(s_n)}$ is a normalized measure with values $M_n^{(s_n)}(D_n^{(s_n)})$, $\forall D_n^{(s_n)} \subseteq \Lambda_n^{(s_n)}$, that are positive operators on a complex separable Hilbert space $\mathcal{H}_n$. On the notion of a POV measure, see, for example, the review section in [20].

4. LHV simulation

Consider a possibility of an LHV simulation of an N-partite correlation experiment described by the $S_1 \times \cdots \times S_N$-setting family

$$\mathcal{E} = \{(s_1, \ldots, s_N) \mid s_1 = 1, \ldots, S_1, \ldots, s_N = 1, \ldots, S_N\}, \tag{24}$$

of N-partite joint measurements with joint probability distributions

$$\left\{P^{(\mathcal{E})}_{(s_1,\ldots,s_N)}, \quad s_1 = 1, \ldots, S_1, \ldots, \quad s_N = 1, \ldots, S_N\right\}. \tag{25}$$

The following notion generalizes to an arbitrary multipartite case the concept of a stochastic hidden variable model, formulated by Fine [7] for a bipartite case with two settings and two outcomes per site.

Definition 4. *An $S_1 \times \cdots \times S_N$-setting family (24) of N-partite joint measurements, with $S_1 + \cdots + S_N > N$ and outcomes of any spectral type, discrete or continuous, admits a local hidden variable*[17] *(LHV) model if all its joint probability distributions (25) admit the factorizable representation of the form*

$$P^{(\mathcal{E})}_{(s_1,\ldots,s_N)}\left(\mathrm{d}\lambda_1^{(s_1)} \times \cdots \times \mathrm{d}\lambda_N^{(s_N)}\right) = \int_\Omega P_1^{(s_1)}\left(\mathrm{d}\lambda_1^{(s_1)} \middle| \omega\right) \cdot \ldots \cdot P_N^{(s_N)}\left(\mathrm{d}\lambda_N^{(s_N)} \middle| \omega\right) \nu_{\mathcal{E}}(\mathrm{d}\omega), \tag{26}$$

in terms of a single probability space[18] $(\Omega, \mathcal{F}_\Omega, \nu_{\mathcal{E}})$ *and conditional probability distributions*[19] $P_1^{(s_1)}(\cdot|\omega), \ldots, P_N^{(s_N)}(\cdot|\omega)$, *defined $\nu_{\mathcal{E}}$-almost everywhere on Ω and such that each $P_n^{(s_n)}(\cdot|\omega)$ depends only on a setting of the corresponding measurement at the nth site.*

If, in addition to (26), some distributions $P_n^{(s_n)}(\cdot|\omega)$ corresponding to different sites are correlated then we refer to such an LHV model as conditional.

If every party observes a finite number of outcomes, for example, each $\Lambda_n^{(s_n)} = \Lambda = \{\lambda_1, \ldots, \lambda_K\}$, then it suffices to verify the validity of representation (26) only for all one-point subsets $\left\{\lambda_{k_1}\right\} \times \cdots \times \left\{\lambda_{k_N}\right\} = \left\{\left(\lambda_{k_1}, \ldots, \lambda_{k_N}\right)\right\} \subset \Lambda^N$.

From the LHV representation (26), it follows that any family (24) of N-partite joint measurements admitting an LHV model satisfies[20] the nonsignaling condition (10). We stress that, in an LHV model of a general type, the probability distribution $\nu_{\mathcal{E}}$ has a purely simulation character and may depend on measurement settings of all (or some) parties. Therefore, a family of N-partite joint measurements admitting a general LHV model *does not need* to be either EPR local or Bell local (see section 3).

In view of representations (19), (26), any $S_1 \times \cdots \times S_N$-setting family (24) of EPR local N-partite joint measurements performed on a classical state π on $(\Theta, \mathcal{F}_\Theta)$ admits the LHV model where the probability space is given by $(\Theta, \mathcal{F}_\Theta, \pi)$ and does not depend on either numbers or settings of parties' measurements. This LHV model is of the special, *classical,* type. From definition 3, it follows that Bell's locality [4, 5] of a multipartite correlation experiment is equivalent to the existence for this experiment of an LHV model of the classical type.

If, however, in an $S_1 \times \cdots \times S_N$-setting family (24) of EPR local N-partite joint measurements, each of joint measurements is performed on a quantum state ρ on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ then, in view of (23), this family does not necessarily admit an LHV model. Possible types

[17] This terminology has been formed historically.

[18] In this triple, ν is a probability distribution on a measurable space $(\Omega, \mathcal{F}_\Omega)$ (see footnote 15). In measure theory, triple $(\Omega, \mathcal{F}_\Omega, \nu)$ called a measure space.

[19] For any subset $D \subseteq \Lambda$, function $P(D|\cdot) : \Omega \to [0, 1]$ is measurable.

[20] The converse of this statement is not, in general, true.

of quantum LHV models and their relation to Werner's notion [8] of an LHV model for a multipartite quantum state are considered in section 5.

Let us now specify the following type of an LHV model.

Definition 5. *An LHV model (26), conditional or unconditional, is called deterministic if there exist measurable functions* $f_{n,s_n}:\Omega\to\Lambda_n^{(s_n)}$ *such that, in representation (26), all conditional probability distributions have the special form*[21]

$$P_n^{(s_n)}\big(D_n^{(s_n)}\big|\omega\big)=\chi_{f_{n,s_n}^{-1}(D_n^{(s_n)})}(\omega),\qquad \forall D_n^{(s_n)}\subseteq\Lambda_n^{(s_n)},\tag{27}$$

$\nu_{\mathcal{E}}$*-almost everywhere on* Ω.

In a deterministic LHV model specified by a probability space $(\Omega,\mathcal{F}_\Omega,\nu_{\mathcal{E}})$, to each variable $\omega\in\Omega$, there corresponds the unique outcome $\lambda_n^{(s_n)}=f_{n,s_n}(\omega)$ for any measurement s_n at an nth site, and all joint distributions $P_{(s_1,\ldots,s_N)}^{(\mathcal{E})}$ have the image form

$$P_{(s_1,\ldots,s_N)}^{(\mathcal{E})}\big(D_1^{(s_1)}\times\cdots\times D_N^{(s_N)}\big)=\nu_{\mathcal{E}}\big(f_{1,s_1}^{-1}(D_1^{(s_1)})\cap\cdots\cap f_{N,s_N}^{-1}(D_N^{(s_N)})\big),\tag{28}$$

for any outcome events $D_1^{(s_1)}\subseteq\Lambda_1^{(s_1)},\ldots,D_N^{(s_N)}\subseteq\Lambda_N^{(s_N)}$. The notion of a deterministic LHV model corresponds to the description of an $S_1\times\cdots\times S_N$-setting multipartite correlation experiment in the frame of the Kolmogorov's model [16].

Let an $S_1\times\cdots\times S_N$-setting family (24) of N-partite joint measurements admit an LHV model specified by a probability space $(\Omega,\mathcal{F}_\Omega,\nu_{\mathcal{E}})$. From the structure of representation (26) and formula (3) it follows

(1) the same LHV model holds for any its $K_1\times\cdots\times K_N$-setting subfamily of N-partite joint measurements, where $K_1\leqslant S_1,\ldots,K_N\leqslant S_N$, and for any $S_{n_1}\times\cdots\times S_{n_M}$-setting family

$$\big\{\big(s_{n_1},\ldots,s_{n_M}\big)\,\big|\,s_{n_1}=1,\ldots,S_{n_1},\ldots,s_{n_M}=1,\ldots,S_{n_M}\big\}\tag{29}$$

of M-partite joint measurements: $1\leqslant n_1<\cdots<n_M\leqslant N, 1\leqslant M<N$, induced by family (24);

(2) for any measurable bounded real-valued functions $\varphi_n^{(s_n)}\big(\lambda_n^{(s_n)}\big), n=1,\ldots,N$, the expected value of their product admits the factorizable representation:

$$\big\langle\varphi_1^{(s_1)}\big(\lambda_1^{(s_1)}\big)\cdot\ldots\cdot\varphi_N^{(s_N)}\big(\lambda_N^{(s_N)}\big)\big\rangle_{\mathcal{E}}=\int\Phi_1^{(s_1)}(\omega)\cdot\ldots\cdot\Phi_N^{(s_N)}(\omega)\nu_{\mathcal{E}}(\mathrm{d}\omega),\tag{30}$$

with $\nu_{\mathcal{E}}$-measurable functions $\Phi_n^{(s_n)}(\omega)=\int\varphi_n^{(s_n)}\big(\lambda_n^{(s_n)}\big)P_n^{(s_n)}\big(\mathrm{d}\lambda_n^{(s_n)}\big|\omega\big)$. In a deterministic LHV model, $\Phi_n^{(s_n)}(\omega)=\big(\varphi_n^{(s_n)}\circ f_{n,s_n}\big)(\omega)$ and, in case of real-valued outcomes,

$$\big\langle\lambda_{n_1}^{(s_{n_1})}\cdot\ldots\cdot\lambda_{n_M}^{(s_{n_M})}\big\rangle_{\mathcal{E}}=\int f_{n_1,s_{n_1}}(\omega)\cdot\ldots\cdot f_{n_M,s_{n_M}}(\omega)\nu_{\mathcal{E}}(\mathrm{d}\omega),\tag{31}$$

where the values of functions f_{n,s_n} constitute outcomes under the corresponding measurements at the corresponding sites.

The following theorem establishes the mutual equivalence of *four* different statements on an LHV simulation of a multipartite correlation experiment. Statements (a)–(c) generalize to an arbitrary multipartite case, with any number of settings and any spectral type of outcomes at each site, the corresponding propositions of Fine [7] for a 2 × 2-setting bipartite case with two outcomes per site. Statement (d) establishes in a general setting the equivalence between the existence of an LHV model (27) and the existence of the LHV-form representation (33) for the product expectations of the special type.

[21] Here, $\chi_{f_{n,s_n}^{-1}(D_n^{(s_n)})}(\omega)$ is an indicator function of the preimage $f_{n,s_n}^{-1}(D_n^{(s_n)})$, see (21).

Theorem 1. *For an $S_1 \times \cdots \times S_N$-setting family (24) of N-partite joint measurements, with any spectral type of outcomes at each site, the following statements are equivalent:*

(a) there exists an LHV model formulated by definition 4;
(b) there exists a deterministic LHV model specified by definition 5;
(c) there exists a joint probability distribution

$$\mu_{\mathcal{E}}\big(\mathrm{d}\lambda_1^{(1)} \times \cdots \times \mathrm{d}\lambda_1^{(S_1)} \times \cdots \times \mathrm{d}\lambda_N^{(1)} \times \cdots \times \mathrm{d}\lambda_N^{(S_N)}\big) \tag{32}$$

that returns all distributions $P_{(s_1,\ldots,s_N)}^{(\mathcal{E})}$ of family (24) as marginals;
(d) there exists a probability space $(\Omega, \mathcal{F}_\Omega, \nu_{\mathcal{E}})$ and $\nu_{\mathcal{E}}$-measurable real-valued functions $\Psi_n^{(s_n)} : \Omega \to [-1, 1]$ on $(\Omega, \mathcal{F}_\Omega)$ such that, for any ± 1-valued functions $\psi_n^{(s_n)} : \Lambda_n^{(s_n)} \to \{-1, 1\}$, the LHV-form representation

$$\big\langle \psi_{n_1}^{(s_{n_1})}\big(\lambda_{n_1}^{(s_{n_1})}\big) \cdot \ldots \cdot \psi_{n_M}^{(s_{n_M})}\big(\lambda_{n_M}^{(s_{n_M})}\big)\big\rangle_{\mathcal{E}} = \int \Psi_{n_1}^{(s_{n_1})}(\omega) \cdot \ldots \cdot \Psi_{n_M}^{(s_{n_M})}(\omega)\nu_{\mathcal{E}}(\mathrm{d}\omega) \tag{33}$$

holds for arbitrary

$$1 \leqslant n_1 < \cdots < n_M \leqslant N, \qquad 1 \leqslant M \leqslant N. \tag{34}$$

Proof. Implication (b) $\Rightarrow$ (a) is obvious and implication (a) $\Rightarrow$ (d) follows from property (30). Let (a) hold. Then each $P_{(s_1,\ldots,s_N)}^{(\mathcal{E})}$ admits representation (26) specified by some probability space $(\Omega', \mathcal{F}_{\Omega'}, \nu'_{\mathcal{E}})$ and conditional distributions $P_n^{(s_n)}(\cdot|\omega')$. The joint probability measure

$$\int_{\Omega'_{\mathcal{E}}} \prod_{s_n,n} P_n^{(s_n)}\big(\mathrm{d}\lambda_n^{(s_n)}\,\big|\,\omega'\big)\nu'_{\mathcal{E}}(\mathrm{d}\omega') \tag{35}$$

on $\Lambda_1^{(1)} \times \cdots \times \Lambda_1^{(S_1)} \times \cdots \times \Lambda_N^{(1)} \times \cdots \times \Lambda_N^{(S_N)}$ returns all distributions $P_{(s_1,\ldots,s_N)}^{(\mathcal{E})}$ of family (24) as marginals. Hence, (a) $\Rightarrow$ (c).

Suppose that (c) holds. Then each $P_{(s_1,\ldots,s_N)}^{(\mathcal{E})}$ represents the corresponding marginal of $\mu_{\mathcal{E}}$ and this means that, for any events $D_n^{(s_n)} \subseteq \Lambda_n^{(s_n)}$,

$$P_{(s_1,\ldots,s_N)}^{(\mathcal{E})}\big(D_1^{(s_1)} \times \cdots \times D_N^{(s_N)}\big) = \int \chi_{D_1^{(s_1)}}\big(\lambda_1^{(s_1)}\big) \cdot \ldots \cdot \chi_{D_N^{(s_N)}}\big(\lambda_N^{(s_N)}\big)\mu_{\mathcal{E}}(\mathrm{d}\lambda_1 \times \cdots \times \mathrm{d}\lambda_N), \tag{36}$$

where, for short, we denote

$$\lambda_n := \big(\lambda_n^{(1)}, \ldots, \lambda_n^{(S_n)}\big), \qquad \Lambda_n := \Lambda_n^{(1)} \times \cdots \times \Lambda_n^{(S_n)}. \tag{37}$$

Representation (36) constitutes a particular case of the LHV representation (26), specified by

$$\begin{aligned} &\omega = (\lambda_1, \ldots, \lambda_N), \qquad \Omega = \Lambda_1 \times \cdots \times \Lambda_N, \\ &\nu_{\mathcal{E}} = \mu_{\mathcal{E}}, \qquad P_n^{(s_n)}\big(D_n^{(s_n)}\,\big|\,\omega\big) = \chi_{D_n^{(s_n)}}\big(\lambda_n^{(s_n)}\big), \end{aligned} \tag{38}$$

and, hence, (c) $\Rightarrow$ (a). Introducing further measurable functions $f_{n,s_n} : \Omega \to \Lambda_n^{(s_n)}$, defined by the relation $f_{n,s_n}(\omega) := \lambda_n^{(s_n)}$, and noting that[22]

$$\chi_{D_n^{(s_n)}}\big(\lambda_n^{(s_n)}\big) = \chi_{f_{n,s_n}^{-1}(D_n^{(s_n)})}(\omega), \tag{39}$$

we represent (36) in the form

$$\begin{aligned} P_{(s_1,\ldots,s_N)}^{(\mathcal{E})}\big(D_1^{(s_1)} \times \cdots \times D_N^{(s_N)}\big) &= \int_\Omega \chi_{f_{1,s_1}^{-1}(D_1^{(s_1)})}(\omega) \cdot \ldots \cdot \chi_{f_{N,s_N}^{-1}(D_N^{(s_N)})}(\omega)\nu_{\mathcal{E}}(\mathrm{d}\omega) \\ &= \nu_{\mathcal{E}}\big(f_{1,s_1}^{-1}\big(D_1^{(s_1)}\big) \cap \cdots \cap f_{N,s_N}^{-1}\big(D_N^{(s_N)}\big)\big). \end{aligned} \tag{40}$$

[22] For notation $f_{n,s_n}^{-1}(D_n^{(s_n)})$, see (21).

This representation for (36) and definition 5 mean that (c) $\Rightarrow$ (b). Thus, we have proved

$$\text{(a)} \Leftrightarrow \text{(b)} \Leftrightarrow \text{(c)}, \qquad \text{(a)} \Rightarrow \text{(d)}, \tag{41}$$

and it remains only to show that (d) implies (a).

Consider ± 1-valued functions $\psi_n^{(s_n)}\big(\lambda_n^{(s_n)}\big) \in \{-1, 1\}$. Let $D_n^{(s_n)} \subseteq \Lambda_n^{(s_n)}$ be a subset where a function $\psi_n^{(s_n)}$ admits the value $(+1)$. The relation

$$\psi_n^{(s_n)}\big(\lambda_n^{(s_n)}\big) = 2\chi_{D_n^{(s_n)}}\big(\lambda_n^{(s_n)}\big) - 1 \tag{42}$$

establishes the one-to-one correspondence between ± 1-valued functions $\psi_n^{(s_n)}$ on $\Lambda_n^{(s_n)}$ and subsets $D_n^{(s_n)} \subseteq \Lambda_n^{(s_n)}$. Due to (42), each ± 1-valued function $\psi_n^{(s_n)}$ on $\Lambda_n^{(s_n)}$ is uniquely specified by a subset $D_n^{(s_n)} \subseteq \Lambda_n^{(s_n)}$, and we replace notation $\psi_n^{(s_n)} \to \psi_{D_n^{(s_n)}}$. Taking (42) into account in representation (4), we derive

$$P_{(s_1,\ldots,s_N)}^{(\mathcal{E})}\big(D_1^{(s_1)} \times \cdots \times D_N^{(s_N)}\big) = \frac{1}{2^N}\big\langle \big\{1 + \psi_{D_1^{(s_1)}}\big(\lambda_1^{(s_1)}\big)\big\} \cdot \ldots \cdot \big\{1 + \psi_{D_N^{(s_N)}}\big(\lambda_N^{(s_N)}\big)\big\}\big\rangle_{\mathcal{E}}. \tag{43}$$

Suppose that (d) holds. Then, for representation (33) it follows that, for each n and each S_n, a correspondence between functions $\psi_{D_n^{(s_n)}}$ and $\Psi_n^{(s_n)}$ is such that $\big(\Psi_n^{(s_n)}\big(\Lambda_n^{(s_n)}\big)\big)(\omega) = 1$ and $\big(\Psi_n^{(s_n)}(\varnothing)\big)(\omega) = -1$, $\nu_{\mathcal{E}}$-almost everywhere on Ω.

Substituting (33) into (43), we derive that any joint distribution $P_{(s_1,\ldots,s_N)}$ admits the LHV representation:

$$P_{(s_1,\ldots,s_N)}^{(\mathcal{E})}\big(D_1^{(s_1)} \times \cdots \times D_N^{(s_N)}\big) = \int_\Omega P_1^{(s_1)}\big(D_1^{(s_1)}\big|\omega\big) \cdot \ldots \cdot P_N^{(s_N)}\big(D_N^{(s_N)}\big|\omega\big)\nu_{\mathcal{E}}(\mathrm{d}\omega), \tag{44}$$

where

$$P_n^{(s_n)}\big(D_n^{(s_n)}\big|\omega\big) = \tfrac{1}{2}\big\{1 + \big(\Psi_n^{(s_n)}\big(D_n^{(s_n)}\big)\big)(\omega)\big\}. \tag{45}$$

Thus, (d) $\Rightarrow$ (a). In view of (41), this proves the mutual equivalence of all statements of theorem 1. □

Since different joint probability measures may have the same marginals, in view of statement (c) of theorem 1, the same multipartite correlation experiment may admit a few LHV models not reducible to each other.

Consider a particular N-partite case where, say, the nth party performs $S_n \geqslant 2$ measurements while all other parties perform only one measurement, $S_k = 1, k \neq n$. Due to reindexing of sites, any of such cases is reduced to the $S_1 \times 1 \times \cdots \times 1$-setting case.

Proposition 2. *For an arbitrary $S_1 \geqslant 2$, any $S_1 \times 1 \times \cdots \times 1$-setting family of N-partite joint measurements satisfying the nonsignaling condition (10) admits an LHV model.*

Proof. For an $S_1 \times 1 \times \cdots \times 1$-setting family $\mathcal{E}$ of N-partite joint measurements, each joint distribution $P_{(s_1,1,\ldots,1)}^{(\mathcal{E})}$, $s_1 \in \{1, \ldots, S_1\}$, satisfies the relation

$$P_{(s_1,1,\ldots,1)}^{(\mathcal{E})}\big(\Lambda_1^{(s_1)} \times D'\big) = 0 \quad \Rightarrow \quad P_{(s_1,1,\ldots,1)}^{(\mathcal{E})}\big(D_1^{(s_1)} \times D'\big) = 0, \tag{46}$$

for any subsets $D_1^{(s_1)} \subseteq \Lambda_1^{(s_1)}$ and $D' \subseteq \Lambda' = \Lambda_2^{(1)} \times \cdots \times \Lambda_N^{(1)}$.

Implication (46) means that, for any subset $D_1^{(s_1)} \subseteq \Lambda_1^{(s_1)}$, the probability distribution $P_{(s_1,1,\ldots,1)}^{(\mathcal{E})}\big(D_1^{(s_1)} \times \mathrm{d}\lambda'\big)$ of outcomes $\lambda' := \big(\lambda_2^{(1)}, \ldots, \lambda_N^{(1)}\big)$ in Λ' is absolutely continuous[23]

[23] On this notion and the Radon–Nikodym theorem, see, for example, [18, 19].

with respect to the marginal $P^{(\mathcal{E})}_{(s_1,1,\ldots,1)}\big(\Lambda_1^{(s_1)} \times \mathrm{d}\lambda'\big)$. Therefore, from the Radon–Nikodym theorem it follows

$$P^{(\mathcal{E})}_{(s_1,1,\ldots,1)}\big(\mathrm{d}\lambda_1^{(s_1)} \times \mathrm{d}\lambda_2^{(1)} \times \cdots \times \mathrm{d}\lambda_N^{(1)}\big) = \alpha^{(\mathcal{E})}_{s_1}\big(\mathrm{d}\lambda_1^{(s_1)}\big|\lambda_2^{(1)},\ldots,\lambda_N^{(1)}\big)P^{(\mathcal{E})}_{(s_1,1,\ldots,1)}\big(\Lambda_1^{(s_1)} \times \mathrm{d}\lambda_2^{(1)} \times \cdots \times \mathrm{d}\lambda_N^{(1)}\big), \tag{47}$$

where $\alpha^{(\mathcal{E})}_{s_1}\big(\mathrm{d}\lambda_1^{(s_1)}\,\big|\,\lambda_2^{(1)},\ldots,\lambda_N^{(1)}\big)$ is a conditional probability distribution of outcomes in $\Lambda_1^{(s_1)}$, given a certain $(\lambda_2^{(1)},\ldots,\lambda_N^{(1)}) \in \Lambda'$. Since all N-partite joint measurements $(s_1, 1, \ldots, 1)$ satisfy the nonsignaling condition (10), we have

$$\begin{aligned} P^{(\mathcal{E})}_{(s_1,1,\ldots,1)}\big(\Lambda_1^{(s_1)} \times \mathrm{d}\lambda_2^{(1)} \times \cdots \times \mathrm{d}\lambda_N^{(1)}\big) &= P^{(\mathcal{E})}_{(s_1',1,\ldots,1)}\big(\Lambda_1^{(s_1')} \times \mathrm{d}\lambda_2^{(1)} \times \cdots \times \mathrm{d}\lambda_N^{(1)}\big) \\ &\equiv \tau^{(\mathcal{E})}\big(\mathrm{d}\lambda_2^{(1)} \times \cdots \times \mathrm{d}\lambda_N^{(1)}\big), \qquad \forall s_1, s_1' \in \{1,\ldots,S_1\}. \end{aligned} \tag{48}$$

The joint probability distribution

$$\big(\alpha_1^{(\mathcal{E})}\big(\mathrm{d}\lambda_1^{(1)}\big|\lambda_2^{(1)},\ldots,\lambda_N^{(1)}\big)\cdot\ldots\cdot\alpha_{S_1}^{(\mathcal{E})}\big(\mathrm{d}\lambda_1^{(S_1)}\big|\lambda_2^{(1)},\ldots,\lambda_N^{(1)}\big)\big)\tau^{(\mathcal{E})}\big(\mathrm{d}\lambda_2^{(1)} \times \cdots \times \mathrm{d}\lambda_N^{(1)}\big) \tag{49}$$

returns all distributions $P^{(\mathcal{E})}_{(s_1,1,\ldots,1)}$, $s_1 = 1,\ldots,S_1$, as the corresponding marginals. In view of implication (c) ⇒ (a) in theorem 1, this proves the statement. □

Consider now an LHV simulation of a bipartite correlation experiment.

Due to proposition 2, for an arbitrary $S_1 \geqslant 2$, any $S_1 \times 1$-setting family of bipartite joint measurements satisfying the nonsignaling condition (10) admits an LHV model. The existence of an LHV model for an arbitrary $S_1 \times S_2$-setting family of bipartite joint measurements is specified by the following theorem[24].

Theorem 2. *Necessary and sufficient condition for an $S_1 \times S_2$-setting family of bipartite joint measurements, with outcomes of any spectral type, to admit an LHV model is the existence of joint probability distributions*[25]*:*

$$\mu_{\blacktriangleright}^{(s_1)}\big(\mathrm{d}\lambda_1^{(s_1)} \times \mathrm{d}\lambda_2^{(1)} \times \cdots \times \mathrm{d}\lambda_2^{(S_2)}\big), \qquad s_1 = 1,\ldots,S_1, \tag{50}$$

such that each $\mu_{\blacktriangleright}^{(s_1)}$ returns all distributions $P^{(\mathcal{E})}_{(s_1,s_2)}$, $s_2 = 1,\ldots,S_2$, as marginals and all $\mu_{\blacktriangleright}^{(s_1)}$, $s_1 = 1,\ldots,S_1$, are compatible in the sense that the relation

$$\mu_{\blacktriangleright}^{(s_1)}\big(\Lambda_1^{(s_1)} \times \mathrm{d}\lambda_2^{(1)} \times \cdots \times \mathrm{d}\lambda_2^{(S_2)}\big) = \mu_{\blacktriangleright}^{(s_1')}\big(\Lambda_1^{(s_1')} \times \mathrm{d}\lambda_2^{(1)} \times \cdots \times \mathrm{d}\lambda_2^{(S_2)}\big) \tag{51}$$

holds for any $s_1, s_1' \in \{1,\ldots,S_1\}$. The same concerns the existence of joint probability distributions

$$\mu_{\blacktriangleleft}^{(s_2)}\big(\mathrm{d}\lambda_1^{(1)} \times \cdots \times \mathrm{d}\lambda_1^{(S_1)} \times \mathrm{d}\lambda_2^{(s_2)}\big), \qquad s_2 = 1,\ldots,S_2, \tag{52}$$

such that each $\mu_{\blacktriangleleft}^{(s_2)}$ returns all distributions $P^{(\mathcal{E})}_{(s_1,s_2)}$, $s_1 = 1,\ldots,S_1$, as marginals and all $\mu_{\blacktriangleleft}^{(s_2)}$, $s_2 = 1,\ldots,S_2$, satisfy the relation

$$\mu_{\blacktriangleleft}^{(s_2)}\big(\mathrm{d}\lambda_1^{(1)} \times \cdots \times \mathrm{d}\lambda_1^{(S_1)} \times \mathrm{d}\lambda_2^{(s_2)}\big) = \mu_{\blacktriangleleft}^{(s_2')}\big(\mathrm{d}\lambda_1^{(1)} \times \cdots \times \mathrm{d}\lambda_1^{(S_1)} \times \mathrm{d}\lambda_2^{(s_2')}\big), \tag{53}$$

for any $s_2, s_2' \in \{1,\ldots,S_2\}$.

[24] This theorem generalizes to an *arbitrary* $S_1 \times S_2$-setting case, with outcomes of any spectral type, Fine's proposition 1 [7, p 292] for the 2×2-setting case with two outcomes per site.

[25] The lower indices of measures $\mu_{\blacktriangleright}^{(s_1)}$ on $\Lambda_1^{(s_1)} \times \Lambda_2^{(1)} \times \cdots \times \Lambda_2^{(S_2)}$ and $\mu_{\blacktriangleleft}^{(s_2)}$ on $\Lambda_1^{(1)} \times \cdots \times \Lambda_1^{(S_1)} \times \Lambda_2^{(s_2)}$ indicate a direction of a direct product extension of set $\Lambda_1^{(s_1)} \times \Lambda_2^{(s_2)}$.

Proof. Denote, for short,

$$\lambda_2 := \big(\lambda_2^{(1)}, \ldots, \lambda_2^{(S_2)}\big), \qquad \Lambda_2 := \Lambda_2^{(1)} \times \cdots \times \Lambda_2^{(S_2)}. \tag{54}$$

For each distribution $\mu_{\blacktriangleright}^{(s_1)}\big(\mathrm{d}\lambda_1^{(s_1)} \times \mathrm{d}\lambda_2\big)$ in (50), the relation

$$\mu_{\blacktriangleright}^{(s_1)}\big(\Lambda_1^{(s_1)} \times D_2\big) = 0 \quad \Rightarrow \quad \mu_{\blacktriangleright}^{(s_1)}\big(D_1^{(s_1)} \times D_2\big) = 0 \tag{55}$$

holds for any subsets $D_1^{(s_1)} \subseteq \Lambda_1^{(s_1)}$ and $D_2 \subseteq \Lambda_2$. This means that, for any $D_1^{(s_1)} \subseteq \Lambda_1^{(s_1)}$, the probability measure $\mu_{\blacktriangleright}^{(s_1)}\big(D_1^{(s_1)} \times \mathrm{d}\lambda_2\big)$ of outcomes in Λ_2 is absolutely continuous[26] with respect to the marginal probability distribution $\mu_{\blacktriangleright}^{(s_1)}\big(\Lambda_1^{(s_1)} \times \mathrm{d}\lambda_2\big)$. Therefore, each $\mu_{\blacktriangleright}^{(s_1)}$ admits the Radon–Nikodym representation:

$$\mu_{\blacktriangleright}^{(s_1)}\big(\mathrm{d}\lambda_1^{(s_1)} \times \mathrm{d}\lambda_2\big) = \alpha_1^{(s_1)}\big(\mathrm{d}\lambda_1^{(s_1)}\big|\lambda_2\big)\mu_{\blacktriangleright}^{(s_1)}\big(\Lambda_1^{(s_1)} \times \mathrm{d}\lambda_2\big), \tag{56}$$

where $\alpha_1^{(s_1)}(\cdot|\lambda_2)$ is a conditional probability distribution of outcomes $\lambda_1^{(s_1)} \in \Lambda_1^{(s_1)}$. In view of (51), we denote

$$\mu_{\blacktriangleright}^{(s_1)}\big(\Lambda_1^{(s_1)} \times \mathrm{d}\lambda_2\big) = \mu_{\blacktriangleright}^{(s_1')}\big(\Lambda_1^{(s_1')} \times \mathrm{d}\lambda_2\big) = \tau_2(\mathrm{d}\lambda_2), \qquad s_1, s_1' \in \{1, \ldots, S_1\}. \tag{57}$$

The joint probability measure

$$\big(\alpha_1^{(1)}\big(\mathrm{d}\lambda_1^{(1)}\big|\lambda_2\big) \cdot \ldots \cdot \alpha_1^{(S_1)}\big(\mathrm{d}\lambda_1^{(S_1)}\big|\lambda_2\big)\big)\tau_2(\mathrm{d}\lambda_2) \tag{58}$$

returns all $P_{(s_1,s_2)}^{(\mathcal{E})}$ as marginals. In view of theorem 1, this proves the sufficiency part of theorem 2.

In order to prove the necessity part, let an $S_1 \times S_2$-setting family admit an LHV model. Then, by statement (c) of theorem 1, there exists a joint probability distribution $\mu_{\mathcal{E}}\big(\mathrm{d}\lambda_1^{(1)} \times \cdots \times \mathrm{d}\lambda_1^{(S_1)} \times \mathrm{d}\lambda_2^{(1)} \times \cdots \times \mathrm{d}\lambda_2^{(S_2)}\big)$ of all outcomes observed by two parties. The marginals

$$\mu_{\mathcal{E}}\big(\Lambda_1^{(1)} \times \cdots \times \Lambda_1^{(s_1-1)} \times \mathrm{d}\lambda_1^{(s_1)} \times \Lambda_1^{(s_1+1)} \times \cdots \times \Lambda_1^{(S_1)} \times \mathrm{d}\lambda_2^{(1)} \times \cdots \times \mathrm{d}\lambda_2^{(S_2)}\big), \tag{59}$$

constitute the probability distributions $\mu_{\blacktriangleright}^{(s_1)}$, specified by (50) and (51). For measures $\mu_{\blacktriangleleft}^{(s_2)}$, the necessity and sufficiency parts are proved quite similarly. □

Theorems 1, 2 and proposition 2 refer to an LHV simulation of an arbitrary multipartite correlation experiment with outcomes of any spectral type. Below, we consider peculiarities of an LHV simulation in a multipartite case with only two outcomes per site.

4.1. A dichotomic multipartite case

Let, under an N-partite joint measurement $(s_1, \ldots, s_N)$, each party perform a measurement with only two outcomes, that is, a *dichotomic* measurement. These two outcomes do not need to be numbers, however, due to possible mappings $\lambda_n^{(s_n)} \mapsto \varphi_n^{(s_n)}\big(\lambda_n^{(s_n)}\big) \in \{-1, 1\}$, it suffices to analyse only a dichotomic case with outcomes, $\lambda_n^{(s_n)} = \pm 1$.

Since the direct product $\big\{\lambda_1^{(s_1)}\big\} \times \cdots \times \big\{\lambda_N^{(s_N)}\big\}$ of one-point subsets constitutes the one-point subset $\big\{(\lambda_1^{(s_1)}, \ldots, \lambda_N^{(s_N)})\big\} \subset \Lambda_1^{(s_1)} \times \cdots \times \Lambda_N^{(s_N)}$, for a discrete case, we further omit brackets $\{\cdot\}$ and denote

$$P_{(s_1,\ldots,s_N)}^{(\mathcal{E})}\big(\big\{\lambda_1^{(s_1)}\big\} \times \cdots \times \big\{\lambda_N^{(s_N)}\big\}\big) \equiv P_{(s_1,\ldots,s_N)}^{(\mathcal{E})}\big(\lambda_1^{(s_1)}, \ldots, \lambda_N^{(s_N)}\big). \tag{60}$$

For a further consideration, we need to prove the following general statement.

[26] See reference in footnote 23.

Lemma 1. *For an arbitrary N-partite joint measurement* $(s_1, \ldots, s_N) \in \mathcal{E}$, *with* ± 1*-valued outcomes at each site,*

$$2P^{(\mathcal{E})}_{(s_1,\ldots,s_N)}\left(\lambda_1^{(s_1)}, \ldots, \lambda_N^{(s_N)}\right) = 1 + \sum_{\substack{1\leqslant n_1<\cdots<n_{N-k}\leqslant N,\\ k=0,\ldots,N-1}} \xi\left(\lambda_{n_1}^{(s_{n_1})}\right)\cdot\ldots\cdot\xi\left(\lambda_{n_{N-k}}^{(s_{n_{N-k}})}\right)\left\langle\lambda_{n_1}^{(s_{n_1})}\cdot\ldots\cdot\lambda_{n_{N-k}}^{(s_{n_{N-k}})}\right\rangle_{\mathcal{E}}, \tag{61}$$

where $\xi(\pm 1) = \pm 1$.

Proof. Due to relations

$$2\chi_{\{1\}}\left(\lambda_n^{(s_n)}\right) - 1 = \lambda_n^{(s_n)}, \qquad 2\chi_{\{-1\}}\left(\lambda_n^{(s_n)}\right) - 1 = -\lambda_n^{(s_n)}, \tag{62}$$

holding for each $\lambda_n^{(s_n)} \in \{-1, 1\}$, we have

$$\chi_{D_n^{(s_n)}}\left(\lambda_n^{(s_n)}\right) = \frac{1+\lambda_n^{(s_n)}\xi\left(D_n^{(s_n)}\right)}{2}, \qquad \xi(\{1\}) = 1, \quad \xi(\{-1\}) = -1, \tag{63}$$

for each of one-point subsets $\{-1\}$ or $\{1\}$.

Substituting (63) into (4), for any direct product combination $D_1^{(s_1)} \times \cdots \times D_N^{(s_N)}$ of one-point subsets $\{-1\}$ and $\{1\}$, we derive

$$\begin{aligned} P^{(\mathcal{E})}_{(s_1,\ldots,s_N)}&\left(D_1^{(s_1)} \times \cdots \times D_N^{(s_N)}\right) \\ &= \frac{1}{2^N}\left\langle\left(1+\lambda_1^{(s_1)}\xi\left(D_1^{(s_1)}\right)\right)\cdot\ldots\cdot\left(1+\lambda_N^{(s_N)}\xi\left(D_N^{(s_N)}\right)\right)\right\rangle_{\mathcal{E}} \\ &= \frac{1}{2^N} + \frac{1}{2^N}\sum_{\substack{1\leqslant n_1<\cdots<n_{N-k}\leqslant N,\\ k=0,\ldots,N-1}} \xi\left(D_{n_1}^{(s_{n_1})}\right)\cdot\ldots\cdot\xi\left(D_{n_{N-k}}^{(s_{n_{N-k}})}\right)\left\langle\lambda_{n_1}^{(s_{n_1})}\cdot\ldots\cdot\lambda_{n_{N-k}}^{(s_{n_{N-k}})}\right\rangle_{\mathcal{E}}. \end{aligned} \tag{64}$$

Using in (64) notation (60) and renaming $\xi(\{1\}) \to \xi(1)$, $\xi(\{-1\}) \to \xi(-1)$, we prove (61). □

From (61) it, in particular, follows:

$$2^N P^{(\mathcal{E})}_{(s_1,\ldots,s_N)}(1, \ldots, 1) = 1 + \sum_{\substack{1\leqslant n_1<\cdots<n_{N-k}\leqslant N,\\ k=0,\ldots,N-1}} \left\langle\lambda_{n_1}^{(s_{n_1})}\cdot\ldots\cdot\lambda_{n_{N-k}}^{(s_{n_{N-k}})}\right\rangle_{\mathcal{E}}. \tag{65}$$

In view of lemma 1, the mutual equivalence of statements (a) and (d) of theorem 1 takes the following form.

Theorem 3. *An* $S_1 \times \cdots \times S_N$*-setting family (24) of N-partite joint measurements, with* ± 1*-valued outcomes at each site, admits an LHV model, formulated by definition 4, iff there exist a probability space* $(\Omega, \mathcal{F}_\Omega, \nu_{\mathcal{E}})$ *and* $\nu_{\mathcal{E}}$*-measurable real-valued functions*

$$f_{n,s_n} : \Omega \to [-1, 1], \qquad \forall s_n, \forall n, \tag{66}$$

on $(\Omega, \mathcal{F}_\Omega)$ *such that any of the mean values*

$$\left\langle\lambda_{n_1}^{(s_{n_1})}\cdot\ldots\cdot\lambda_{n_M}^{(s_{n_M})}\right\rangle_{\mathcal{E}}, \qquad 1 \leqslant n_1 < \cdots < n_M \leqslant N, \qquad 1 \leqslant M \leqslant N, \tag{67}$$

admits the representation

$$\left\langle\lambda_{n_1}^{(s_{n_1})}\cdot\ldots\cdot\lambda_{n_M}^{(s_{n_M})}\right\rangle_{\mathcal{E}} = \int f_{n_1,s_{n_1}}(\omega)\cdot\ldots\cdot f_{n_M,s_{n_M}}(\omega)\nu_{\mathcal{E}}(\mathrm{d}\omega) \tag{68}$$

of the LHV form.

Proof. The necessity follows from property 2 (see formula (30)). In order to prove the sufficiency part, let us substitute (68) into formula (61), in the form (64). For any direct product combination $D_1^{(s_1)} \times \cdots \times D_N^{(s_N)}$ of one-point subsets $\{-1\}$ and $\{1\}$, we derive

$$P^{(\mathcal{E})}_{(s_1,\ldots,s_N)}\big(D_1^{(s_1)} \times \cdots \times D_N^{(s_N)}\big) = \frac{1}{2^N}\int \big[1+\xi\big(D_1^{(s_1)}\big)f_{1,s_1}(\omega)\big]\cdot\ldots\cdot\big[1+\xi\big(D_N^{(s_N)}\big)f_{N,s_N}(\omega)\big]\nu_{\mathcal{E}}(\mathrm{d}\omega). \tag{69}$$

Extending (69) to all subsets of set $\{-1, 1\}$, we have

$$P^{(\mathcal{E})}_{(s_1,\ldots,s_N)}\big(D_1^{(s_1)} \times \cdots \times D_N^{(s_N)}\big) = \int P_1^{(s_1)}\big(D_1^{(s_1)}\big|\omega\big)\cdot\ldots\cdot P_N^{(s_N)}\big(D_N^{(s_N)}\big|\omega\big)\nu_{\mathcal{E}}(\mathrm{d}\omega), \tag{70}$$

where

$$\begin{aligned} &P_n^{(s_n)}(\{1\}|\omega) = \tfrac{1}{2}[1+f_{n,s_n}(\omega)], \qquad P_n^{(s_n)}(\{-1\}|\omega) = \tfrac{1}{2}[1-f_{n,s_n}(\omega)],\\ &P_n^{(s_n)}(\varnothing|\omega) = 0, \qquad P_n^{(s_n)}(\{-1,1\}|\omega) = 1. \end{aligned} \tag{71}$$

This proves the statement. □

From theorem 3 it follows that, for an arbitrary $S_1 \times \cdots \times S_N$-setting family of N-partite joint measurements with two outcomes per site, the existence of the LHV-form representation (68) *only* for the full correlation functions *does not*, in general, imply the existence of an LHV model (26) for joint probability distributions.

All statements of section 4 refer to an LHV simulation of a general correlation experiment. In the following section, we specify an LHV simulation in a quantum multipartite case.

5. Quantum LHV models

We start by analysing an LHV simulation of an $S_1 \times S_2$-setting family of *EPR local* bipartite joint measurements performed on a separable quantum state:

$$\rho_{\mathrm{sep}} = \sum_m \gamma_m \rho_1^{(m)} \otimes \rho_2^{(m)}, \qquad \gamma_m \geqslant 0, \qquad \sum_m \gamma_m = 1, \tag{72}$$

on a complex separable Hilbert space $\mathcal{H} \otimes \mathcal{H}$, possibly, infinite dimensional.

Let, at each nth site, quantum measurements be described by POV measures $M_n^{(s_n)}\big(\mathrm{d}\lambda_n^{(s_1)}\big)$, $s_n = 1, \ldots, S_n, n = 1, 2$. From (23) and (72) it follows that this correlation experiment is described by the joint probability distributions of the form

$$P_{(s_1,s_2)}\big(\mathrm{d}\lambda_1^{(s_1)} \times \mathrm{d}\lambda_2^{(s_2)}\big|\rho_{\mathrm{sep}}\big) = \sum_m \gamma_m \mathrm{tr}\big[\rho_1^{(m)} M_1^{(s_1)}\big(\mathrm{d}\lambda_1^{(s_1)}\big)\big]\,\mathrm{tr}\big[\rho_2^{(m)} M_2^{(s_2)}\big(\mathrm{d}\lambda_2^{(s_2)}\big)\big]. \tag{73}$$

This form constitutes a particular case of the LHV representation (26), specified by the probability space with elements

$$\Omega' = \{m = 1, 2, \ldots\}, \qquad \nu'_m = \gamma_m, \qquad \forall m \in \Omega', \tag{74}$$

and conditional distributions $P_n^{(s_n)}(\cdot|m) = \mathrm{tr}\big[\rho_n^{(m)} M_n^{(s_n)}(\cdot)\big]$, $s_n = 1, \ldots, S_n, n = 1, 2$, for any $m \in \Omega'$.

Thus, any $S_1 \times S_2$-setting family of bipartite joint measurements performed on a separable quantum state ρ_{sep} admits the LHV model where the probability space is determined only by this separable state and does not depend on either numbers or settings of parties' measurements, that is, the LHV model of the classical type (see section 4).

Furthermore, all $P^{(\mathcal{E})}_{(s_1,s_2)}(\cdot|\rho_{\text{sep}}), s_1 = 1, \ldots, S_1, s_2 = 1, \ldots, S_2$, defined by (73), are marginals of the joint probability measure:

$$\mu_{\rho_{\text{sep}}}\left(\mathrm{d}\lambda_1^{(1)} \times \cdots \times \mathrm{d}\lambda_1^{(S_1)} \times \mathrm{d}\lambda_2^{(1)} \times \cdots \times \mathrm{d}\lambda_2^{(S_2)}\right) = \sum_m \gamma_m \prod_{s_1,s_2} \mathrm{tr}\left[\rho_1^{(m)} M_1^{(s_1)}\left(\mathrm{d}\lambda_1^{(s_1)}\right)\right] \mathrm{tr}\left[\rho_2^{(m)} M_2^{(s_2)}\left(\mathrm{d}\lambda_2^{(s_2)}\right)\right]. \tag{75}$$

Therefore, from the proof of implication (c) $\Rightarrow$ (a) in theorem 1 (see representation (36)) it follows that the considered correlation experiment also admits the LHV model which is specified by the probability space $(\Omega, \mathcal{F}_\Omega, \mu_{\rho_{\text{sep}}})$, with

$$\begin{aligned} \omega &= \left(\lambda_1^{(1)}, \ldots, \lambda_1^{(S_1)}, \lambda_2^{(1)}, \ldots, \lambda_2^{(S_2)}\right), \\ \Omega &= \Lambda_1^{(1)} \times \cdots \times \Lambda_1^{(S_1)} \times \Lambda_2^{(1)} \times \cdots \times \Lambda_2^{(S_2)}, \end{aligned} \tag{76}$$

and conditional distributions $P_n^{(s_n)}\left(D_n^{(s_n)} \mid \omega\right) = \chi_{D_n^{(s_n)}}(\omega)$. The latter LHV model is induced by the LHV model (74).

Consider further an $S_1 \times S_2$-setting bipartite correlation experiment, performed on the *specific* bipartite separable state

$$\widetilde{\rho}_{\text{sep}} = \sum_m \gamma_m |e_m\rangle\langle e_m| \otimes |e_m\rangle\langle e_m|, \tag{77}$$

where $\{e_m\}$ is an orthonormal basis in $\mathcal{H}$. Since state $\widetilde{\rho}_{\text{sep}}$ is reduced from the nonseparable pure state

$$T = \left|\sum_m \sqrt{\gamma_m} e_m^{\otimes(S_1+S_2)}\right\rangle\left\langle\sum_m \sqrt{\gamma_m} e_m^{\otimes(S_1+S_2)}\right| \tag{78}$$

on $\mathcal{H}^{\otimes(S_1+S_2)}$, all distributions $P_{(s_1,s_2)}(\cdot|\widetilde{\rho}_{\text{sep}})$ represent marginals of the joint measure

$$\begin{aligned} \mu'_{\widetilde{\rho}_{\text{sep}}}&\left(\mathrm{d}\lambda_1^{(1)} \times \cdots \times \mathrm{d}\lambda_1^{(S_1)} \times \mathrm{d}\lambda_2^{(1)} \times \cdots \times \mathrm{d}\lambda_2^{(S_2)}\right) \\ &= \mathrm{tr}\left[T\left\{M_1^{(1)}\left(\mathrm{d}\lambda_1^{(1)}\right) \otimes \cdots \otimes M_1^{(S_1)}\left(\mathrm{d}\lambda_1^{(S_1)}\right) \otimes M_2^{(1)}\left(\mathrm{d}\lambda_2^{(1)}\right) \otimes \cdots \otimes M_2^{(S_2)}\left(\mathrm{d}\lambda_2^{(S_2)}\right)\right\}\right] \\ &= \sum_{m,l} \sqrt{\gamma_m}\sqrt{\gamma_l} \prod_{s_1,s_2} \langle e_m|M_1^{(s_1)}\left(\mathrm{d}\lambda_1^{(s_1)}\right)|e_l\rangle\langle e_m|M_2^{(s_2)}\left(\mathrm{d}\lambda_2^{(s_2)}\right)|e_l\rangle. \end{aligned} \tag{79}$$

Quite similarly as explained above, this implies that any $S_1 \times S_2$-setting family of bipartite joint measurements performed on $\widetilde{\rho}_{\text{sep}}$ admits the LHV model, specified by the probability space $\left(\Omega, \mathcal{F}_\Omega, \mu'_{\widetilde{\rho}_{\text{sep}}}\right)$, where variables $\omega \in \Omega$ are defined by (76) while distribution $\mu_{\widetilde{\rho}_{\text{sep}}} \neq \mu'_{\widetilde{\rho}_{\text{sep}}}$. The latter LHV model is not reducible to the LHV model (74) of the classical type.

Thus, any $S_1 \times S_2$-setting bipartite correlation experiment performed on state $\widetilde{\rho}_{\text{sep}}$ admits at least two LHV models not reducible to each other. The first LHV model, with the probability space (74) depending only on state $\widetilde{\rho}_{\text{sep}}$, holds for *any setting* $S_1 \times S_2$. The second LHV model, with the probability space $\left(\Omega, \mathcal{F}_\Omega, \mu'_{\widetilde{\rho}_{\text{sep}}}\right)$, is constructed specifically for a given setting $S_1 \times S_2$.

In view of this analysis, we introduce the following notions.

Definition 6. *An N-partite quantum state ρ admits an $S_1 \times \cdots \times S_N$-setting LHV description if any $S_1 \times \cdots \times S_N$-setting family of EPR local N-partite joint measurements performed on this quantum state admits an LHV model formulated by definition 4.*

This definition and the LHV property 1 (specified in section 3 after definition 5) imply the following statements on an LHV description of an arbitrary N-partite quantum state.

Proposition 3. *Let an N-partite quantum state ρ on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ admit an $S_1 \times \cdots \times S_N$-setting LHV description. Then, (i) ρ admits any $K_1 \times \cdots \times K_N$-setting LHV description where*

$K_1 \leqslant S_1, \ldots, K_N \leqslant S_N$; (ii) for any sites $1 \leqslant n_1 < \cdots < n_M \leqslant N$, where $1 \leqslant M < N$, the reduced M-partite state $\rho_{(n_1,\ldots,n_M)}$ on $\mathcal{H}_{n_1} \otimes \cdots \otimes \mathcal{H}_{n_M}$ admits the $S_{n_1} \times \cdots \times S_{n_M}$-setting LHV description.

We stress that an N-partite quantum state ρ, admitting the $K_1 \times \cdots \times K_N$-setting LHV description, does not need to admit an $S_1 \times \cdots \times S_N$-setting LHV description with $S_1 > K_1, \ldots, S_N > K_N$.

Definition 7. *An N-partite quantum state ρ is said to admit an LHV model of Werner's type if any setting family of EPR local N-partite joint measurements performed on this state admits one and the same LHV model formulated by definition 4.*

Any separable state admits an LHV model of Werner's type. For a bipartite case, this model is specified by (74). The nonseparable Werner state [8] $W_{d,\Phi}$ on $\mathbb{C}^d \otimes \mathbb{C}^d$, $d \geqslant 2$, with parameter $\Phi \geqslant -1 + \frac{d+1}{d^2}$, admits [8] an LHV model of Werner's type under any projective measurements of two parties.

From definitions 6 and 7 it follows that if an N-partite quantum state ρ admits an LHV model of Werner's type then it admits an LHV description for any setting $S_1 \times \cdots \times S_N$. However, the converse of this statement is not true and even if an N-partite quantum state ρ admits an LHV description for any setting $S_1 \times \cdots \times S_N$, this does not imply that this ρ admits an LHV model of Werner's type—since for each concrete setting $S_1 \times \cdots \times S_N$, a probability space may depend not only a state ρ but also on performed measurements.

From definition 6 and proposition 2 in section 3 it follows the following statement.

Proposition 4. *An arbitrary N-partite quantum state ρ admits an $S_1 \times \underbrace{1 \times \cdots \times 1}_{N-1}$-setting LHV description for any $S_1 \geqslant 2$.*

Consider now a convex combination of N-partite quantum states admitting an LHV description for a definite $S_1 \times \cdots \times S_N$ setting.

Proposition 5. *Let each of quantum states $\rho_1, \ldots, \rho_M$ on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ admit an $S_1 \times \cdots \times S_N$-setting LHV description. Then, any of their convex combinations*

$$\sum_m \gamma_m \rho_m, \qquad \gamma_m \geqslant 0, \qquad \sum_m \gamma_m = 1, \tag{80}$$

also admits the $S_1 \times \cdots \times S_N$-setting LHV description.

Proof. Suppose that every state ρ_m on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ admits an $S_1 \times \cdots \times S_N$-setting LHV description. Then, by definition 6 and theorem 1, for any $S_1 \times \cdots \times S_N$-setting family of N-partite joint measurements (23), performed on ρ_m and specified by POV measures $M_n^{(s_n)}$, $\forall s_n, \forall n$, there exists a joint probability distribution

$$\mu_m\big(\mathrm{d}\lambda_1^{(1)} \times \cdots \times \mathrm{d}\lambda_1^{(S_1)} \times \cdots \times \mathrm{d}\lambda_N^{(1)} \times \cdots \times \mathrm{d}\lambda_N^{(S_N)}\big), \tag{81}$$

returning all

$$P_{(s_1,\ldots,s_N)}\big(\mathrm{d}\lambda_1^{(s_1)} \times \cdots \times \mathrm{d}\lambda_N^{(s_N)} \big| \rho_m\big) = \mathrm{tr}\big[\rho_m\big\{M_1^{(s_1)}\big(\mathrm{d}\lambda_1^{(s_1)}\big) \otimes \cdots \otimes M_N^{(s_N)}\big(\mathrm{d}\lambda_N^{(s_N)}\big)\big\}\big],$$
$$s_1 = 1, \ldots, S_1, \ldots, s_N = 1, \ldots, S_N, \tag{82}$$

as marginals. This implies that, for a mixture $\eta = \sum_m \gamma_m \rho_m$, every

$$P_{(s_1,\ldots,s_N)}\big(\mathrm{d}\lambda_1^{(s_1)} \times \cdots \times \mathrm{d}\lambda_N^{(s_N)} \big| \eta\big)$$
$$= \sum_m \gamma_m \, \mathrm{tr}\big[\rho_m\big\{M_1^{(s_1)}\big(\mathrm{d}\lambda_1^{(s_1)}\big) \otimes \cdots \otimes M_N^{(s_N)}\big(\mathrm{d}\lambda_N^{(s_N)}\big)\big\}\big] \tag{83}$$

constitutes the corresponding marginal of distribution $\sum_m \gamma_m \mu_m$. Therefore, by item (c) of theorem 1, any $S_1 \times \cdots \times S_N$-setting family of N-partite joint measurements on state $\sum_m \gamma_m \rho_m$ admits an LHV model. By definition 6, the latter means that state η_β admits the $S_1 \times \cdots \times S_N$-setting LHV description. □

In the following statement, proved in the appendix, we establish a threshold bound for an arbitrary noisy bipartite state to admit an $S_1 \times S_2$-setting LHV description. In an $S_1 \times 1$-setting (or $1 \times S_2$-setting) case, this bound is consistent with the statement of proposition 4.

Proposition 6. *Let a bipartite quantum state ρ on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, $d_1, d_2 \geqslant 2$, do not admit the LHV description for a given setting $S_1 \times S_2$. The noisy state*

$$\eta_\rho(\gamma) = (1-\gamma)\frac{I_{\mathbb{C}^{d_1}\otimes\mathbb{C}^{d_2}}}{d_1 d_2} + \gamma\rho, \qquad 0 \leqslant \gamma \leqslant (1+\beta_\rho)^{-1}, \tag{84}$$

admits the $S_1 \times S_2$-setting LHV description under any generalized EPR local quantum measurements of two parties. In (84),

$$\beta_\rho = \min\left\{d_1(S_2-1)\left\|\tau_\rho^{(1)}\right\|; d_2(S_1-1)\left\|\tau_\rho^{(2)}\right\|\right\} \tag{85}$$

and $\left\|\tau_\rho^{(1)}\right\|, \left\|\tau_\rho^{(2)}\right\|$ are operator norms of the reduced states $\tau_\rho^{(1)} = \mathrm{tr}_{\mathbb{C}^{d_2}}[\rho]$ and $\tau_\rho^{(2)} = \mathrm{tr}_{\mathbb{C}^{d_1}}[\rho]$ on $\mathbb{C}^{d_1}$ and $\mathbb{C}^{d_2}$, respectively.

As an example, let us specify bound (84) for the noisy state

$$\eta_\psi^{(d)}(\gamma) = (1-\gamma)\frac{I_{\mathbb{C}^{d}\otimes\mathbb{C}^{d}}}{d^2} + \gamma|\psi\rangle\langle\psi|, \tag{86}$$

on $\mathbb{C}^d \otimes \mathbb{C}^d$, $d \geqslant 2$, induced by the maximally entangled pure state $\psi = \frac{1}{\sqrt{d}}\sum_{m=1}^{d} e_m \otimes e_m$, where $\{e_m\}$ is an orthonormal basis in $\mathbb{C}^d$.

In this case, $\left\|\tau^{(n)}_{|\psi\rangle\langle\psi|}\right\| = \frac{1}{d}, n = 1, 2$, and substituting this into (84), we conclude that state $\eta^{(d)}_{|\psi\rangle\langle\psi|}(\gamma)$ admits an $S_1 \times S_2$-setting LHV description under any *generalized* quantum measurements of two parties whenever

$$0 \leqslant \gamma \leqslant \frac{1}{1 + \min_{n=1,2}(S_n - 1)}. \tag{87}$$

Note that the partial transpose of $\eta^{(d)}_{|\psi\rangle\langle\psi|}(\gamma)$ has the eigenvalue $\frac{1-\gamma(d+1)}{d^2}$, which is negative for any $\gamma > \frac{1}{d+1}$. Therefore, due to the Peres separability criterion [21], state $\eta^{(d)}_{|\psi\rangle\langle\psi|}(\gamma)$ is nonseparable for any $\gamma \in \left(\frac{1}{d+1}, 1\right]$. Thus, for state (86), bound (87) is nontrivial whenever $\min_{n=1,2}(S_n - 1) < d$.

6. Conclusions

In the present paper, we introduce a general framework for the probabilistic description of a multipartite correlation scenario with an arbitrary number of settings and any spectral type of outcomes at each site. This allows us

- To specify in probabilistic terms the difference between nonsignaling [17], the EPR locality [1] and Bell's locality [4, 5] and to show that, in contrast to the opinion of Bell [3, 5]:
  (i) the EPR paradox [1] cannot be, in principle, resolved via the violation of Bell's locality since the latter type of locality is only sufficient but not necessary for the type of locality meant by Einstein, Podolsky and Rosen in [1]—the EPR locality;

(ii) the EPR locality is not violated[27] under a multipartite correlation experiment on spacelike separated quantum particles and the so-called 'quantum nonlocality' does not constitute the violation of locality of quantum interactions;

- To introduce the notion of an LHV model for an $S_1 \times \cdots \times S_N$-setting N-partite correlation experiment with outcomes of any spectral type, discrete or continuous, and to stress that the same correlation experiment may admit several LHV models and that the existence of an LHV model of a general type is necessarily linked with only nonsignaling, but does not need to imply the EPR locality and even Bell's locality;
- To prove general statements on an LHV simulation of an arbitrary $S_1 \times \cdots \times S_N$-setting N-partite correlation experiment. These statements not only generalize to an arbitrary multipartite case, with outcomes of any spectral type, discrete or continuous, the necessary and sufficient conditions introduced by Fine [7] for a 2×2-setting case, with two outcomes per site, but also establish the equivalence between the existence of an LHV model for joint probability distributions and the existence of the LHV-form representation for the product expectations of the special type;
- To introduce the notion of an N-partite quantum state admitting an $S_1 \times \cdots \times S_N$-setting LHV description; to prove the main general statements on this notion and to establish its relation to Werner's concept [8] of an LHV model for a multipartite quantum state;
- To evaluate a threshold visibility for an arbitrary noisy bipartite quantum state to admit an $S_1 \times S_2$-setting LHV description.

In the sequel [25] to this paper, for an $S_1 \times \cdots \times S_N$-setting N-partite correlation experiment with outcomes of any spectral type, discrete or continuous, we introduce a single general representation incorporating in a unique manner all Bell-type inequalities (on either joint probabilities or correlation functions) that have been introduced in the literature ever since the seminal publication [4] of Bell on the original Bell inequality.

Appendix

Consider the proof of proposition 6 in section 5. For the 2×2-setting case, this proof is similar to our proof of theorem 1 in [22].

According to definition 6, in order to prove that state $\eta_\rho(\gamma)$ admits an $S_1 \times S_2$-setting LHV description, we need to show that any $S_1 \times S_2$-setting family of bipartite joint quantum measurements performed on $\eta_\rho(\gamma)$ admits an LHV model.

Let, at each site, quantum measurements be described by POV measures $M_1^{(s_1)}(\mathrm{d}\lambda_1^{(s_1)})$, $s_1 = 1, \ldots, S_1$, and $M_2^{(s_2)}(\mathrm{d}\lambda_2^{(s_2)})$, $s_2 = 1, \ldots, S_2$. From formula (23) it follows that distributions $P_{(s_1,.s_2)}(\cdot|\eta_\rho)$ have the form

$$P_{(s_1,.s_2)}\big(\mathrm{d}\lambda_1^{(s_1)} \times \mathrm{d}\lambda_2^{(s_2)}|\eta_\rho\big) = \mathrm{tr}\big[\eta_\rho\big\{M_1^{(s_1)}\big(\mathrm{d}\lambda_1^{(s_1)}\big) \otimes M_2^{(s_2)}\big(\mathrm{d}\lambda_2^{(s_2)}\big)\big\}\big], \qquad s_1 = 1, \ldots, S_1, \qquad s_2 = 1, \ldots, S_2. \tag{A.1}$$

For state η_ρ on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, introduce self-adjoint operators $T_\blacktriangleright$ on $\mathbb{C}^{d_1} \otimes (\mathbb{C}^{d_2})^{\otimes S_2}$ and $T_\blacktriangleleft$ on $(\mathbb{C}^{d_1})^{\otimes S_1} \otimes \mathbb{C}^{d_2}$, satisfying the relations

$$\mathrm{tr}_{\mathbb{C}^{d_2}}^{(k_1,\ldots,k_{S_2-1})}[T_\blacktriangleright] = \eta_\rho, \qquad 2 \leqslant k_1 < \cdots < k_{S_2-1} \leqslant 1 + S_2, \tag{A.2}$$

$$\mathrm{tr}_{\mathbb{C}^{d_1}}^{(j_1,\ldots,j_{S_1-1})}[T_\blacktriangleleft] = \eta_\rho, \qquad 1 \leqslant j_1 < \cdots < j_{S_1-1} \leqslant S_1. \tag{A.3}$$

Here, (i) the lower indices of operators $T_\blacktriangleright$ and $T_\blacktriangleleft$ indicate the direction of extension of the Hilbert space $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$; (ii) $\mathrm{tr}_{\mathbb{C}^{d_2}}^{(k_1,\ldots,k_{S_2-1})}[\cdot]$ denotes the partial trace over elements of $\mathbb{C}^{d_2}$,

[27] See also our discussion in [12].

standing in k_1th, ..., k_{S_2-1}th places in tensor products in $\mathbb{C}^{d_1} \otimes (\mathbb{C}^{d_2})^{\otimes S_2}$. Similarly, for the partial trace $\mathrm{tr}_{\mathbb{C}^{d_1}}^{(j_1,\ldots,j_{S_1-1})}[\cdot]$.

As we prove in [24], for any bipartite quantum state, dilations $T_\blacktriangleright$ and $T_\blacktriangleleft$ exist. In [23, 24], we refer to these dilations as source operators for a bipartite state. Note that any positive source operator is a density operator.

If, for state $\eta_\rho(\gamma)$, there exist *density source operators* $T_\blacktriangleright$ and $T_\blacktriangleleft$, then the probability measures

$$\mathrm{tr}\big[T_\blacktriangleright\big\{M_1^{(s_1)}\big(\mathrm{d}\lambda_1^{(s_1)}\big) \otimes M_2^{(1)}\big(\mathrm{d}\lambda_2^{(1)}\big) \otimes \cdots \otimes M_2^{(S_2)}\big(\mathrm{d}\lambda_2^{(S_2)}\big)\big\}\big], \qquad s_1 = 1, \ldots, S_1, \tag{A.4}$$

and

$$\mathrm{tr}\big[T_\blacktriangleleft\big\{M_1^{(1)}\big(\mathrm{d}\lambda_1^{(1)}\big) \otimes \cdots \otimes M_1^{(S_1)}\big(\mathrm{d}\lambda_1^{(S_1)}\big) \otimes M_2^{(s_2)}\big(\mathrm{d}\lambda_2^{(s_2)}\big)\big\}\big], \qquad s_2 = 1, \ldots, S_2, \tag{A.5}$$

constitute, correspondingly, distributions

$$\mu_\blacktriangleright^{(s_1)}\big(\mathrm{d}\lambda_1^{(s_1)} \times \mathrm{d}\lambda_2^{(1)} \times \cdots \times \mathrm{d}\lambda_2^{(S_2)}\big), \qquad s_1 = 1, \ldots, S_1, \tag{A.6}$$

and

$$\mu_\blacktriangleleft^{(s_2)}\big(\mathrm{d}\lambda_1^{(1)} \times \cdots \times \mathrm{d}\lambda_1^{(S_1)} \times \mathrm{d}\lambda_2^{(s_2)}\big), \qquad s_2 = 1, \ldots, S_2, \tag{A.7}$$

specified in theorem 2 of section 4.

Therefore, finding for state $\eta_\rho(\gamma)$ of a density source operator $T_\blacktriangleright$ (or $T_\blacktriangleleft$) will prove the existence for this state of an $S_1 \times S_2$-setting LHV description.

For a state ρ standing in (84), consider its spectral decomposition:

$$\rho = \sum_i \alpha_i |\Psi_i\rangle\langle\Psi_i|, \qquad \langle\Psi_i, \Psi_j\rangle = \delta_{ij}, \qquad \forall \alpha_i > 0, \qquad \sum_i \alpha_i = 1. \tag{A.8}$$

Let

$$\Psi_i = \sum_k \Phi_k^{(i)} \otimes f_k, \qquad \sum_k \big\langle\Phi_k^{(i)}, \Phi_k^{(j)}\big\rangle = \delta_{ij}, \tag{A.9}$$

be the Schmidt decomposition of eigenvector Ψ_i with respect to an orthonormal basis $\{f_k\}$ in $\mathbb{C}^{d_2}$. Substituting this into (A.8), we thus derive

$$\rho = \sum_{k,l=1}^{d_2} \rho_{kl} \otimes |f_k\rangle\langle f_l|, \qquad \rho_{kl} = \sum_i \alpha_i \big|\Phi_k^{(i)}\big\rangle\big\langle\Phi_l^{(i)}\big|. \tag{A.10}$$

Operators ρ_{kk} are positive with $\sum_k \mathrm{tr}[\rho_{kk}] = 1$. Note that $\tau_\rho^{(1)} = \sum_k \rho_{kk}$ is the state on $\mathbb{C}^{d_1}$ reduced from ρ.

Introduce on $\mathbb{C}^{d_1} \otimes (\mathbb{C}^{d_2})^{\otimes S_2}$ the self-adjoint operator

$$\begin{aligned} T_\blacktriangleright(\gamma) = (1-\gamma)\frac{I_{\mathbb{C}^{d_1}} \otimes I_{\mathbb{C}^{d_2}} \otimes I_{\mathbb{X}}}{d_1 d_2^{S_2}} + \gamma \sum_{k,l} \rho_{kl} \otimes \frac{\{|f_k\rangle\langle f_l| \otimes I_{\mathbb{X}}\}_{\mathrm{sym}}}{d_2^{S_2-1}} \\ - \gamma(S_2-1)\tau_\rho^{(1)} \otimes \frac{I_{\mathbb{C}^{d_2}} \otimes I_{\mathbb{X}}}{d_2^{S_2}}, \end{aligned} \tag{A.11}$$

where $\mathbb{X} := (\mathbb{C}^{d_2})^{\otimes(S_2-1)}$; operator $\{|f_k\rangle\langle f_l| \otimes I_{\mathbb{X}}\}_{\mathrm{sym}}$ on $(\mathbb{C}^{d_2})^{\otimes S_2}$ represents the symmetrization of $|f_k\rangle\langle f_l| \otimes I_{\mathbb{X}}$ and operators ρ_{kl} on $\mathbb{C}^{d_1}$ are defined by (A.10). It is easy to verify that $T_\blacktriangleright^{(1,S_2)}(\gamma)$ satisfies condition (A.2) and, therefore, constitutes a source operator for state $\eta_\rho(\gamma)$.

In order to find γ for which operator $T_\blacktriangleright(\gamma)$ is positive, let us evaluate the sum of the first and the third terms standing in (A.11). Note that, in view of (A.10), the second term in (A.11) constitutes a positive operator.

Taking into account the relation

$$-\|Y\| I_{\mathcal{K}} \leqslant Y \leqslant \|Y\| I_{\mathcal{K}}, \tag{A.12}$$

holding for any bounded quantum observable Y on a Hilbert space $\mathcal{K}$, we derive

$$(1-\gamma)\frac{I_{\mathbb{C}^{d_1}} \otimes I_{\mathbb{C}^{d_2}} \otimes I_{\mathbb{X}}}{d_1 d_2^{S_2}} - \gamma(S_2-1)\tau_\rho^{(1)} \otimes \frac{I_{\mathbb{C}^{d_2}} \otimes I_{\mathbb{X}}}{d_2^{S_2}}$$
$$\geqslant \left[1-\gamma\left(1+d_1(S_2-1)\left\|\tau_\rho^{(1)}\right\|\right)\right]\frac{I_{\mathbb{C}^{d_1}} \otimes I_{\mathbb{C}^{d_2}} \otimes I_{\mathbb{X}}}{d_1 d_2^{S_2}}. \tag{A.13}$$

Therefore, the source operator $T_{\blacktriangleright}(\gamma)$ is positive (i.e. a density source operator) for any

$$0 \leqslant \gamma \leqslant \left(1+d_1(S_2-1)\left\|\tau_\rho^{(1)}\right\|\right)^{-1}. \tag{A.14}$$

Quite similarly, state $\eta_\rho(\gamma)$ has a density source operator $T_{\blacktriangleleft}(\gamma)$ for any

$$0 \leqslant \gamma \leqslant \left(1+d_2(S_1-1)\left\|\tau_\rho^{(2)}\right\|\right)^{-1}. \tag{A.15}$$

From (A.14) and (A.15) it follows that state $\eta_\rho(\gamma)$ admits an $S_1 \times S_2$-setting LHV description for any

$$0 \leqslant \gamma \leqslant \left(1+\min\left\{d_1(S_2-1)\left\|\tau_\rho^{(1)}\right\|; d_2(S_1-1)\left\|\tau_\rho^{(2)}\right\|\right\}\right). \tag{A.16}$$

Note that $\left\|\tau_\rho^{(1)}\right\| \geqslant \frac{1}{d_1}$ and $\left\|\tau_\rho^{(2)}\right\| \geqslant \frac{1}{d_2}$.

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